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Convex Analysis Approach to Discrete Optimization, III

Algorithms for Discrete Convex Functions

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Algorithms for Discrete Convex Functions

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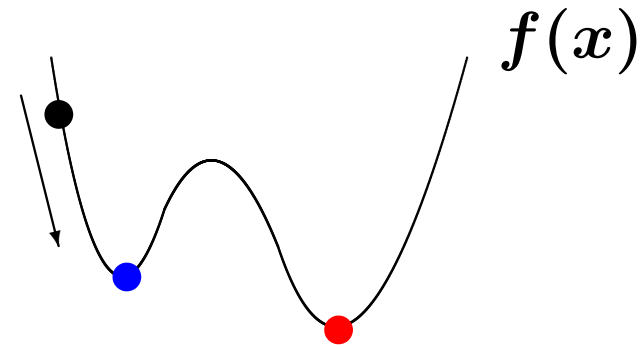
A3. L-convex Minimization

A4. M-convex Intersection

A1.

Minimization (General)

Descent Method



S0: Initial sol x^*

S1: Minimize $f(x)$ in **nbhd** of x^* to obtain x^\bullet

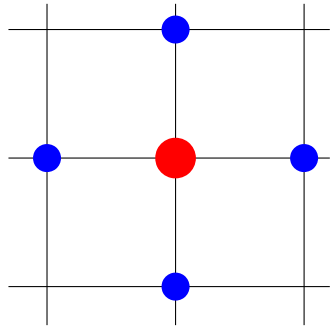
S2: If $f(x^*) \leq f(x^\bullet)$, return x^* (**local opt**)

S3: Update $x^* := x^\bullet$; go to S1

What is **neighborhood**?

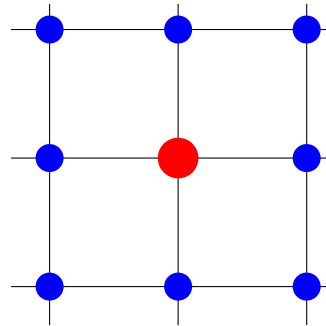
Neighborhood for Local Optimality

separable
convex



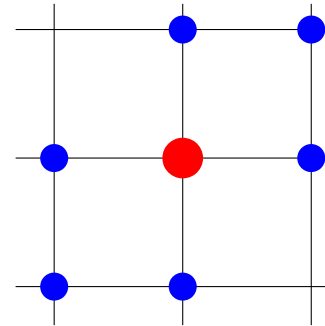
$$2n + 1$$

integrally
convex



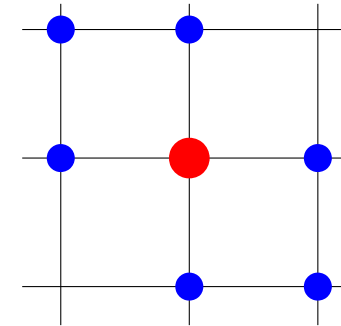
$$3^n$$

L^{\square} -
convex



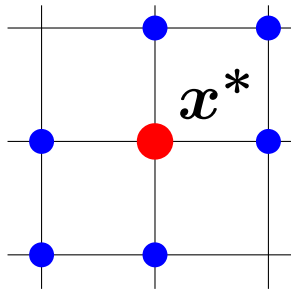
$$2^{n+1} - 1$$

M^{\square} -
convex



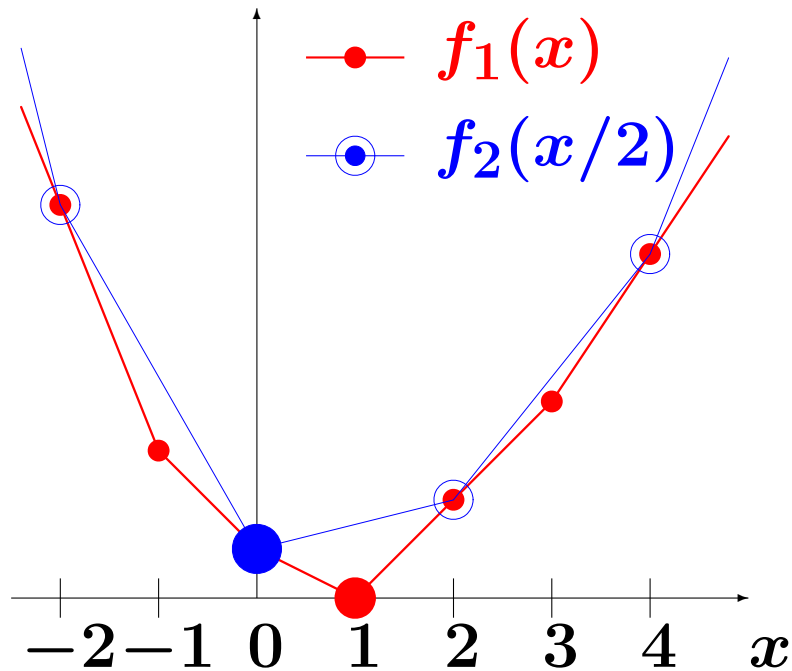
$$n(n + 1) + 1$$

Local Optimality



	#neigh -bors	poly-time local opt	algorithm global opt
submodular (set fn)	2^n	Y	
separable-conv	$2n$	Y	
integrally-conv	3^n	N	
L^{\natural} -conv (\mathbb{Z}^n)	2^n	Y	
M^{\natural} -conv (\mathbb{Z}^n)	n^2	Y	

Scaling and Proximity



Proximity theorem:

True minimum ● exists

in a neighborhood of

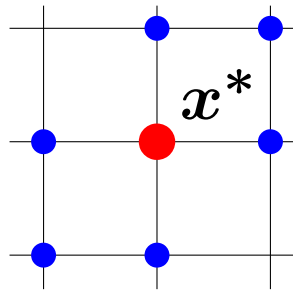
a scaled local minimum ●

⇒ efficient algorithm

Facts in DCA:

- Scaling preserves L-convexity
- Scaling does NOT preserve M-convexity
- Proximity thms known for L-conv and M-conv

Minimization



	#neigh -bors	poly-time local opt	algorithm global opt
submodular (set fn)	2^n	Y	Y
separable-conv	$2n$	Y	Y
integrally-conv	3^n	N	N
L^{\natural} -conv (\mathbb{Z}^n)	2^n	Y	Y
M^{\natural} -conv (\mathbb{Z}^n)	n^2	Y	Y

A2.

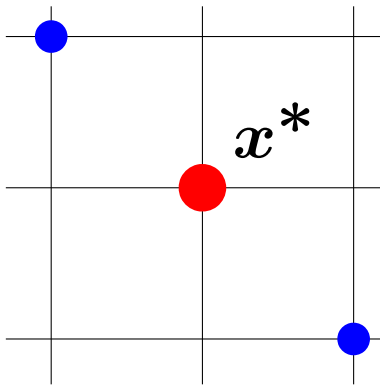
M-convex Minimization

Local vs Global Opt (M-conv)

Thm : $f : \mathbb{Z}^n \rightarrow \mathbb{R}$ **M-convex** (Murota 96)

x^* : global opt

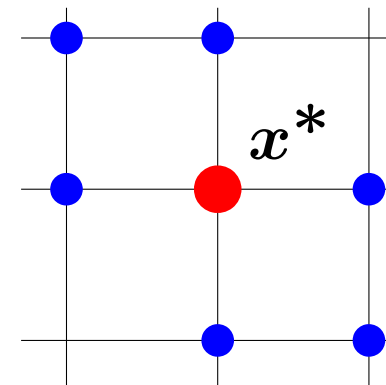
\iff local opt $f(x^*) \leq f(x^* - e_i + e_j)$ $(\forall i, j)$



Ex: $x^* + (0, 1, 0, 0, -1, 0, 0, 0)$

Can check with n^2 fn evals

For M ∇ -convex fn \Rightarrow



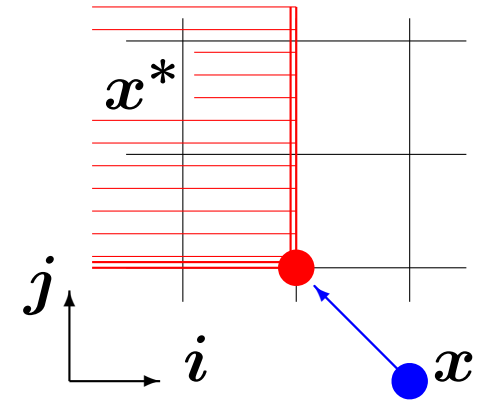
Steepest Descent for M-convex Fn

S0: Find a vector $x \in \text{dom } f$

S1: Find (i, j) that minimizes $f(x - e_i + e_j)$

S2: If $f(x) \leq f(x - e_i + e_j)$, stop
(x : minimizer)

S3: Set $x := x - e_i + e_j$
and go to S1



Minimizer Cut Thm

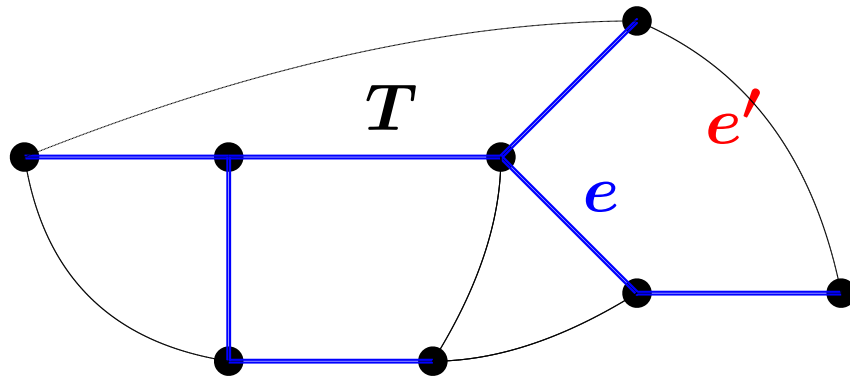
(Shioura 98)

\exists minimizer x^* with $x^*(i) \leq x(i) - 1$, $x^*(j) \geq x(j) + 1$

\Rightarrow Murota 03, Shioura 98, 03, Tamura 05

- Dress–Wenzel’s alg for valuated matroid
- Kalaba’s alg for min spanning tree

Min Spanning Tree Problem



edge length $d : E \rightarrow \mathbb{R}$
total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Thm

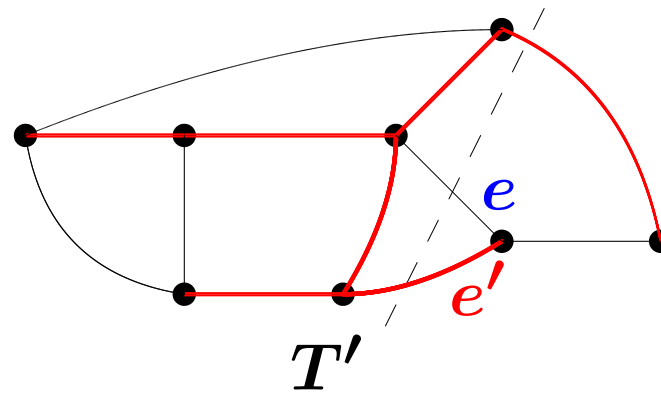
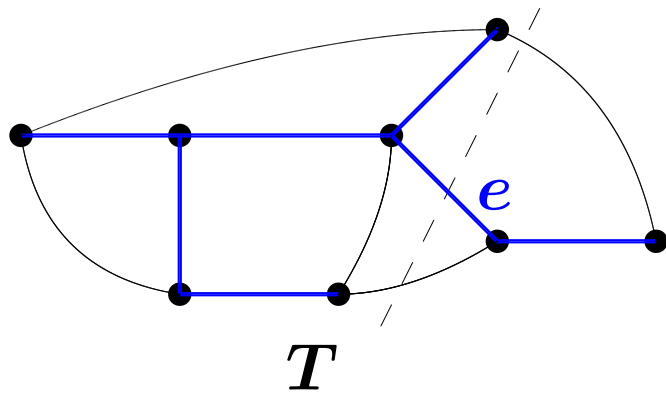
$$\begin{aligned} T: \text{MST} &\iff \tilde{d}(T) \leq \tilde{d}(T - e + e') \\ &\iff d(e) \leq d(e') \quad \text{if } T - e + e' \text{ is tree} \end{aligned}$$

Algorithm Kruskal's, Kalaba's

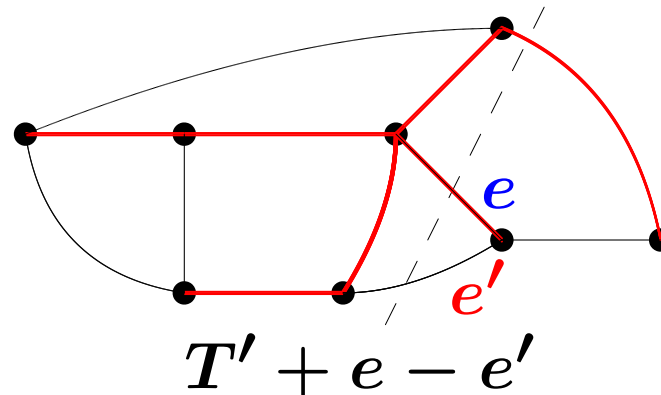
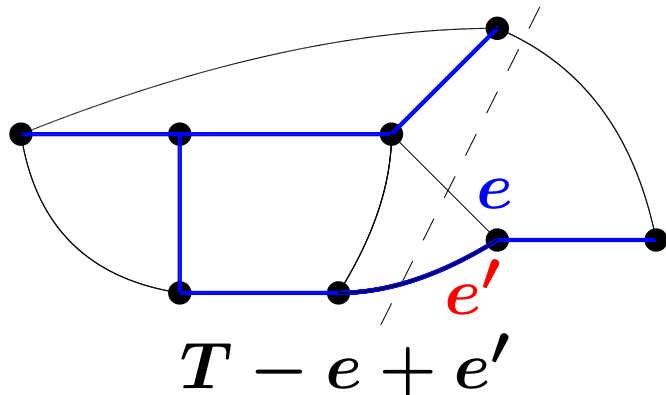
DCA view

- linear optimization on an M-convex set
- M-optimality: $f(x^*) \leq f(x^* - e_i + e_j)$

Tree: Exchange Property



Given pair
of trees



New pair
of trees

Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$
there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

Kruskal's Greedy Algorithm for MST

Kruskal (1959)

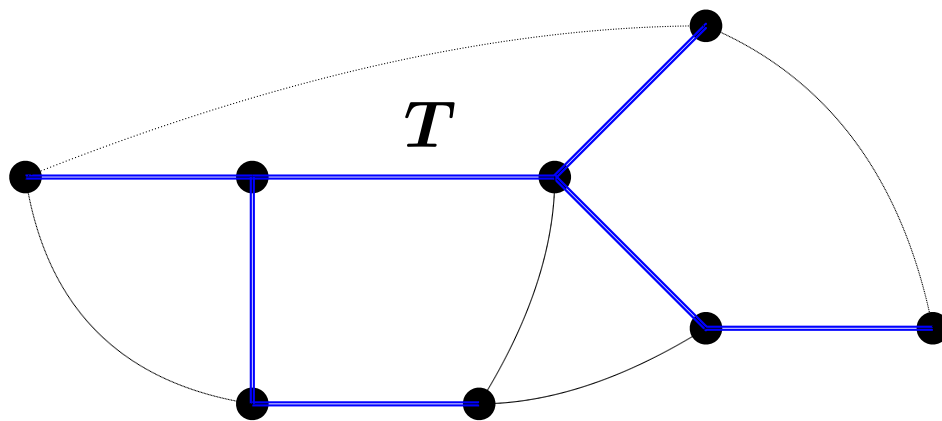
S0: Order edges by length: $d(e_1) \leq d(e_2) \leq \dots$

S1: $T = \emptyset$; $i = 1$

S2: Pick edge e_i

S3: If $T + e_i$ contains a cycle, discard e_i

S4: Update $T = T + e_i$; $i = i + 1$; go to S2



Kalaba's Algorithm for MST

Kalaba (1960), Dijkstra (1960)

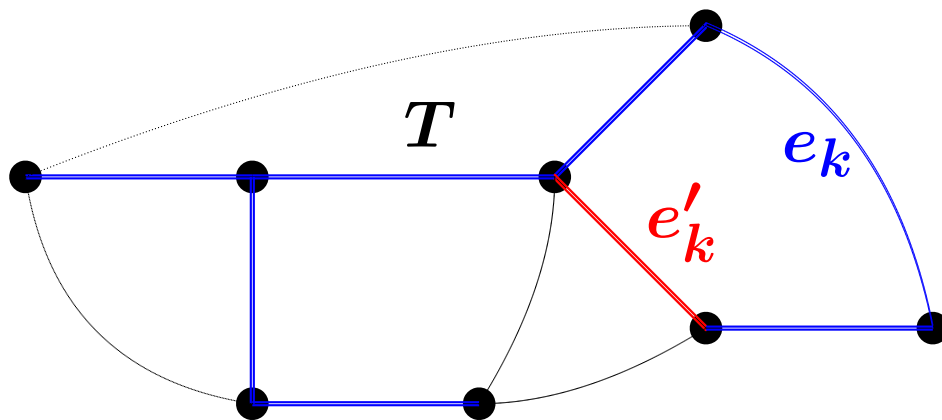
S0: $T =$ any spanning tree

S1: Order $e' \notin T$ by length: $d(e'_1) \leq d(e'_2) \leq \dots$

$k = 1$

S2: $e_k =$ longest edge s.t. $T - e_k + e'_k$ is tree

S3: $T = T - e_k + e'_k$; $k = k + 1$; go to S2



A3.

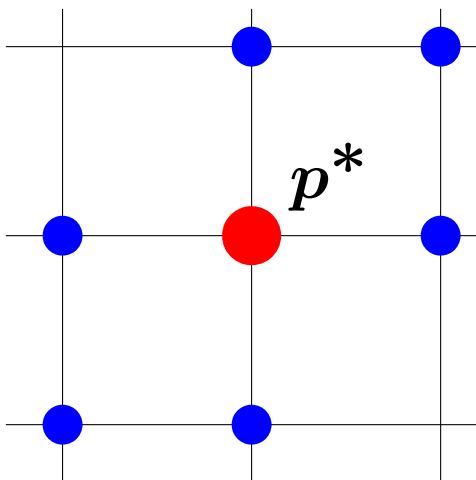
L-convex Minimization

Local vs Global Opt (L^{\natural} -conv)

Thm : $g : \mathbb{Z}^n \rightarrow \mathbb{R}$ L^{\natural} -convex (Murota 98,03)

p^* : global opt

\iff local opt $g(p^*) \leq g(p^* \pm q)$ ($\forall q \in \{0, 1\}^n$)



Ex: $p^* + (0, 1, 0, 1, 1, 1, 0, 0)$

$\iff \rho_{\pm}(X) = g(p^* \pm \chi_X) - g(p^*)$

takes min at $X = \emptyset$

Can check with n^5 (or less) fn evals
using submodular fn min algorithm
(Iwata-Fleischer-Fujishige, Schrijver, Orlin,
Lee-Sidford-Wong)

Steepest Descent for L^{\natural} -convex Fn

(Murota 00, 03, Kolmogorov-Shioura 09, Murota-Shioura 14)

S0: Find a vector $p^{\circ} \in \text{dom } g$ and set $p := p^{\circ}$

S1: Find $\varepsilon = \pm 1$ and X that minimize $g(p + \varepsilon\chi_X)$

S2: If $g(p) \leq g(p + \varepsilon\chi_X)$, stop (p : minimizer)

S3: Set $p := p + \varepsilon\chi_X$ and go to S1

Thm:

(Murota-Shioura 14)

Termination exactly in $\mu(p^{\circ}) + 1$ iterations, where

$$\mu(p^{\circ}) = \min\{\|p^* - p^{\circ}\|_{\infty}^+ + \|p^* - p^{\circ}\|_{\infty}^- \mid p^* \in \arg \min g\}$$

$$\|q\|_{\infty}^+ = \max_i \max(0, q(i)), \quad \|q\|_{\infty}^- = \max_i \max(0, -q(i))$$

Monotone Steepest Descent for L^1 -convex Fn

S0: Find a vector $p^\circ \in \text{dom } g$ s.t

$\{q \mid q \geq p^\circ\} \cap \text{argmin } g \neq \emptyset$ and set $p := p^\circ$

S1: Find X that minimizes $g(p + \chi_X)$

S2: If $g(p) \leq g(p + \chi_X)$, stop (p : minimizer)

S3: Set $p := p + \chi_X$ and go to S1

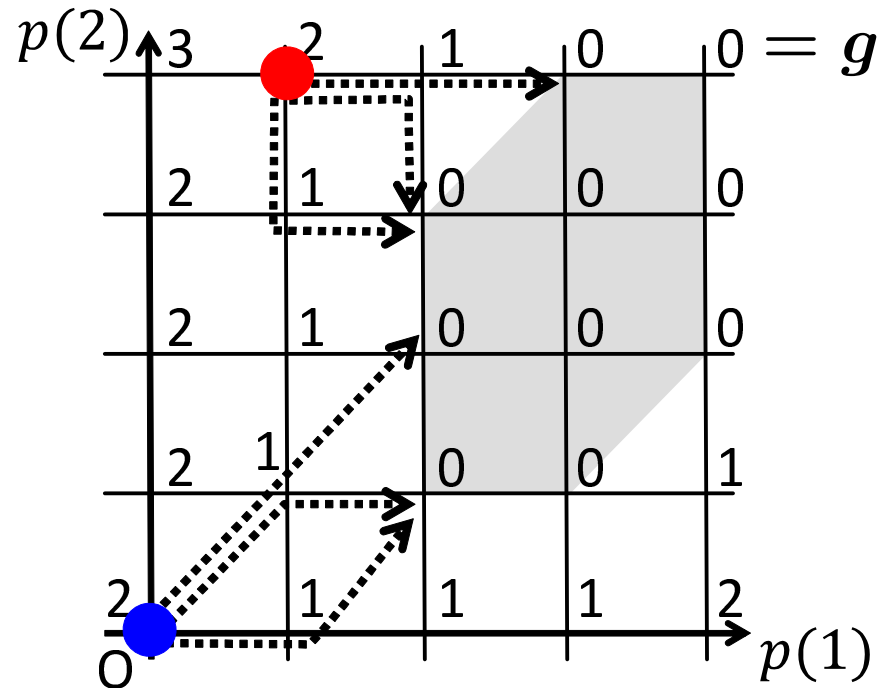
Thm: (Murota-Shioura 14)

Termination exactly in $\hat{\mu}(p^\circ) + 1$ iterations, where

$$\hat{\mu}(p^\circ) = \min\{\|p^* - p^\circ\|_\infty \mid p^* \in \text{argmin } g, p^* \geq p^\circ\}$$

\Rightarrow Application to ascending auction

Steepest Descent Path for L_1 -convex Fn



$$\mu(\mathbf{p}^\circ) = \hat{\mu}(\mathbf{p}^\circ) = 2$$

$$\|\mathbf{p}^\circ, \operatorname{argmin} g\|_\infty = 1$$

$$\mu(\mathbf{p}^\circ) = \hat{\mu}(\mathbf{p}^\circ) = 2$$

$$\|\mathbf{p}^\circ, \operatorname{argmin} g\|_\infty = 2$$

Shortest Path Problem (one-to-all)

one vertex (s) to all vertices, length $\ell \geq 0$, integer

Dual LP

$$\begin{aligned} & \text{Maximize } \sum p(v) \\ & \text{subject to } p(v) - p(u) \leq \ell(u, v) \quad \forall (u, v) \\ & \qquad \qquad p(s) = 0 \end{aligned}$$

Algorithm Dijkstra's

DCA view

- linear optimization on an L^{\natural} -convex set (in polyhedral description)
- Dijkstra's algorithm (Murota-Shioura 12)
= steepest ascent for L^{\natural} -concave maximization
with uniform linear objective $(1, 1, \dots, 1)$

Optimality & Proximity Theorems

Func Class	Optimality	Proximity
L-convex	$f(x^*) \leq f(x^* + \chi_S) \quad (\forall S)$ $f(x^* + 1) = f(x^*) \quad (\text{M. 01})$	$\ x^* - x^\alpha\ \leq (n-1)(\alpha-1)$ (Iwata-Shigeno 03)
M-convex	$f(x^*) \leq f(x^* - \chi_u + \chi_v)$ $(\forall u, v \in V) \quad (\text{M. 96})$	$\ x^* - x^\alpha\ \leq (n-1)(\alpha-1)$ (Moriguchi-M.-Shioura 02)
L2-convex (L□L convol)	$f(x^*) \leq f(x^* + \chi_S) \quad (\forall S)$ $f(x^* + 1) = f(x^*)$	$\ x^* - x^\alpha\ \leq 2(n-1)(\alpha-1)$ (M.-Tamura 04)
M2-convex (M+M)	$f(x^*) \leq f(x^* - \chi_U + \chi_W)$ $(\forall U, W) \quad (\text{M. 01})$	$\ x^* - x^\alpha\ \leq \frac{n^2}{2}(\alpha-1)$ (M.-Tamura 04)
integrally convex	$f(x^*) \leq f(x^* - \chi_U + \chi_W)$ $(\forall U, W) \quad (\text{Favati-Tardella 90})$	$\ x^* - x^\alpha\ \leq \frac{(n+1)!}{2^{n-1}}(\alpha-1)$ (Moriguchi-M.-Tamura -Tardella 16)

$$\| \cdot \| = \| \cdot \|_\infty$$

A4.

M-convex Intersection

(Fenchel Duality)

Intersection Problem $(f_1 + f_2)$

Recall: $L^{\natural} + L^{\natural} \Rightarrow L^{\natural}, \quad M^{\natural} + M^{\natural} \not\Rightarrow M^{\natural}$

M-convex Intersection Algorithm:

Minimizes $f_1 + f_2$ for M^{\natural} -convex f_1, f_2

\Leftrightarrow Maximizes $f_1 + f_2$ for M^{\natural} -concave f_1, f_2
(submodular function maximization)

\Leftrightarrow Fenchel duality (min = max)

\Rightarrow Valuated matroid intersection (Murota 96)

\Rightarrow Weighted matroid intersection

(Edmonds, Lawler, Iri-Tomizawa 76, Frank 81)

M-convex Intersection: Min $[M^\natural + M^\natural]$

$M^\natural + M^\natural$ is NOT M^\natural

$f_1, f_2 : M^\natural$ -convex $(\mathbb{Z}^n \rightarrow \mathbb{R})$, $x^* \in \text{dom } f_1 \cap \text{dom } f_2$

(1) x^* minimizes $f_1 + f_2$ (Murota 96)

$\iff \exists p$ (certificate of optimality)

• x^* minimizes $f_1(x) - \langle p, x \rangle$ (M-opt thm)

• x^* minimizes $f_2(x) + \langle p, x \rangle$ (M-opt thm)

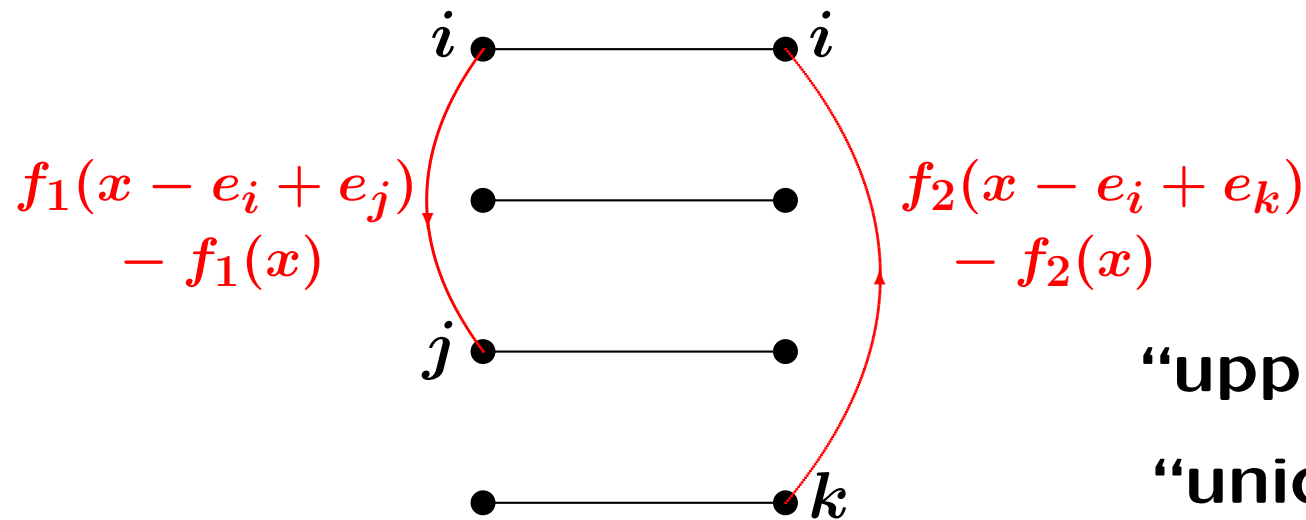
(2) $\text{argmin}(f_1 + f_2) = \text{argmin}(f_1 - p) \cap \text{argmin}(f_2 + p)$

(3) f_1, f_2 are integer-valued \Rightarrow integral p

M-convex Intersection Algorithms

Natural extensions of
weighted (poly)matroid intersection algorithms

Exchange arcs are weighted



“upper-bound lemma”

“unique-max lemma”

- cycle-canceling (Murota 96, 99)
- successive shortest path (Murota-Tamura 03)
- scaling (Iwata-Shigeno 03, Iwata-Moriguchi-Murota 05)

Convolution

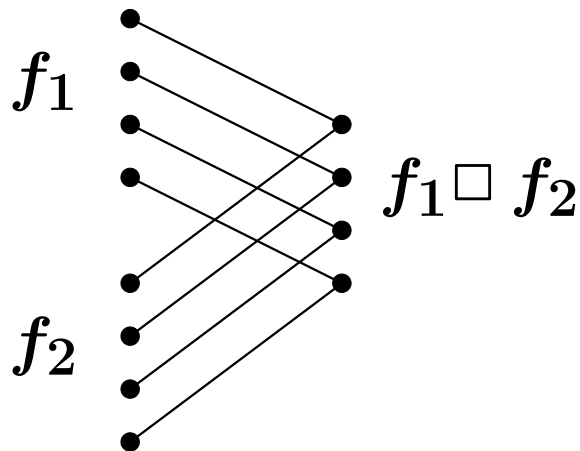
Convolutions of M^{\natural} -convex functions:

$$(f_1 \square f_2)(x) = \min_y (f_1(y) + f_2(x - y))$$

$$(f_1 \square f_2 \square f_3)(x), \quad (f_1 \square f_2 \square \cdots \square f_k)(x)$$

can be computed by **M-convex intersection algorithms**

cf. aggregated utility function



E N D