ICCOPT 2016, Summer School, Tokyo, August 7, 2016

Convex Analysis Approach to Discrete Optimization, II

Properties of Discrete Convex Functions

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Properties of Discrete Convex Functions

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P1.

Convex Extension

Convex Extension

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is convex-extensible

$\Leftrightarrow \exists$ convex $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$: $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$

Theorem:

(1) Separable-convex fns are convex-extensible

(2) Integrally-convex fns are convex-extensible (by def)

(3) $\mathrm{L}^\natural$-convex fns are convex-extensible (Murota 98)

(4) $\mathrm{M}^\natural$-convex fns are convex-extensible (Murota 96)

Convex Extension — Computation

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is convex-extensible

$\Leftrightarrow$ $\exists$ convex $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$: $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z}^n)$

Theorem:

(1) Separable-convex

easy to compute (consecutive points)

(2) Integrally-convex

difficult (exp-time) to compute

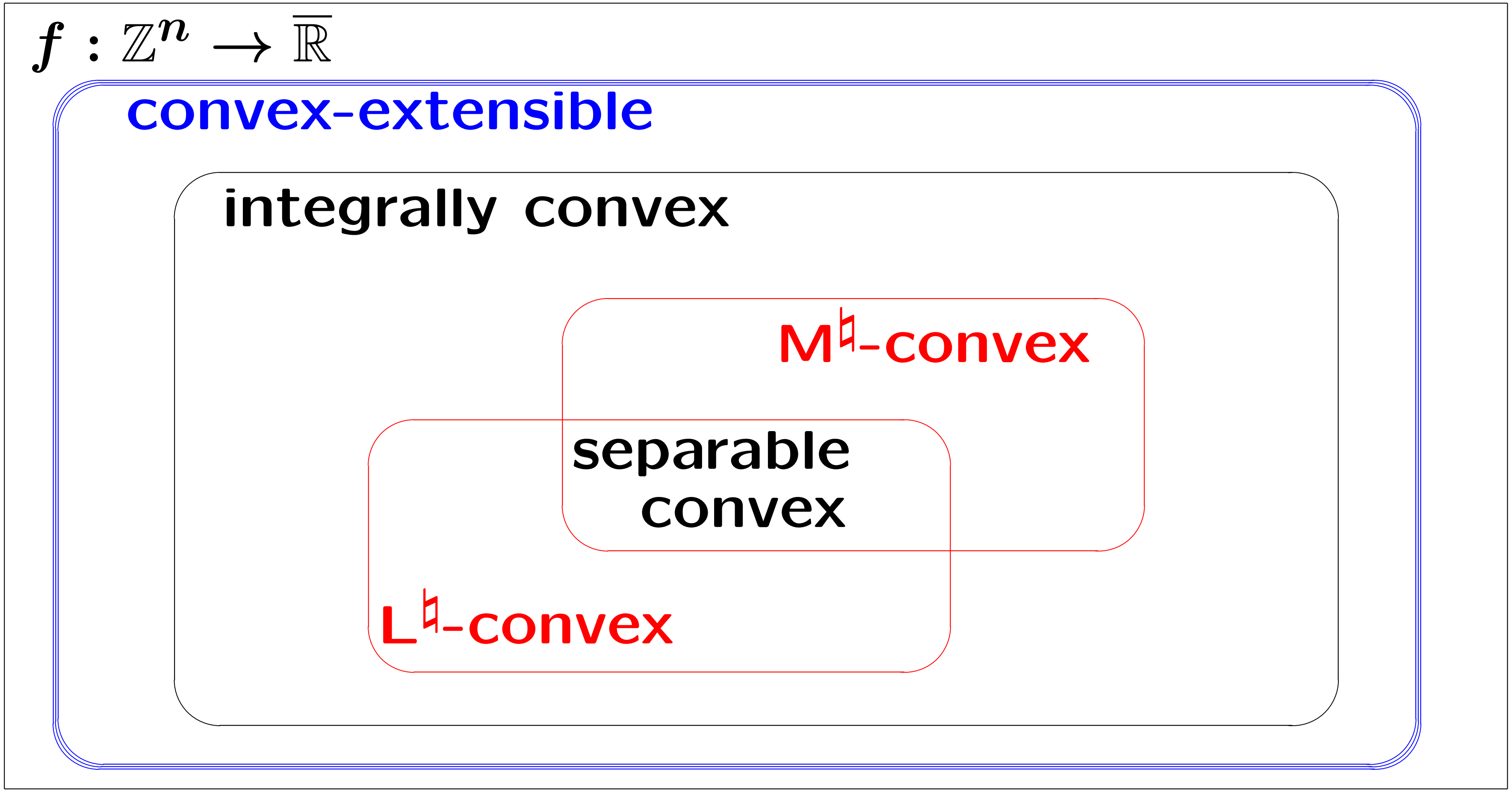
(3) $\mathrm{L}^{\natural}$-convex (Favati–Tardella 90, Murota 98)

easy to compute (Lovász ext.)

(4) $\mathrm{M}^{\natural}$-convex (Shioura 09,15)

poly-time to compute (via conjugacy)

Classes of Discrete Convex Functions



M♮ ∩ L♮ = separable

P2.
Optimality Criterion

(local opt = global opt)

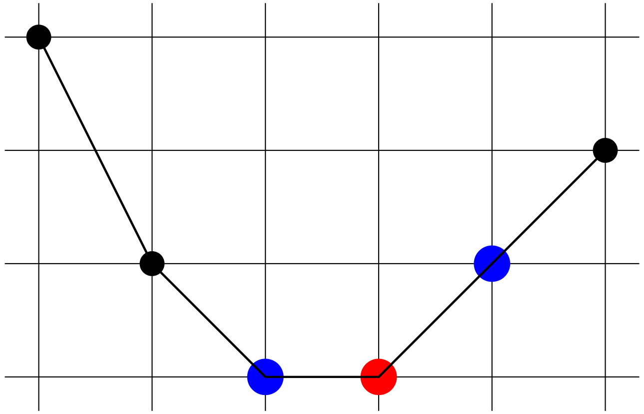
Local vs Global Optimality ($n = 1$)

$f : \mathbb{Z} \to \overline{\mathbb{R}}$

x^*: **global opt (min)**

$\iff$ x^*: **local opt (min)**

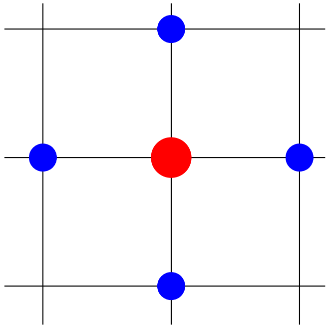
$$f(x^*) \leq \min\{f(x^* - 1),\ f(x^* + 1)\}$$



x^*

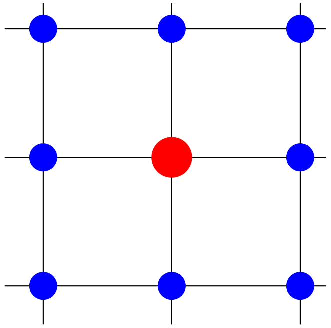
Neighborhood for Local Optimality

separable convex



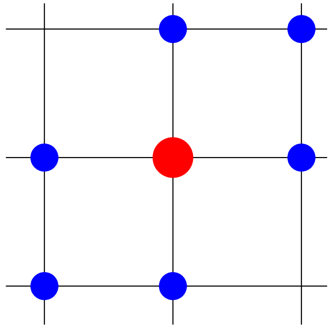
$$2n+1$$

integrally convex



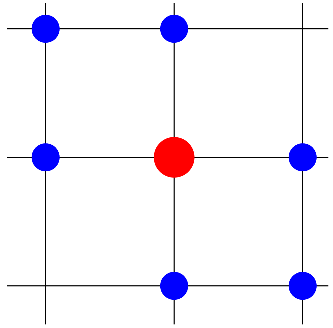
$$3^n$$

L$^\natural$-convex



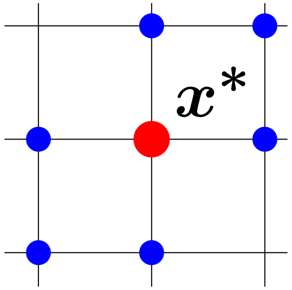
$$2^{n+1}-1$$

M$^\natural$-convex



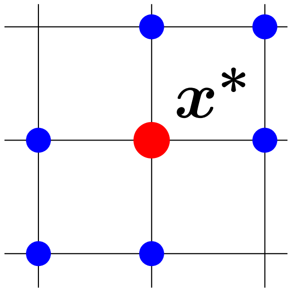
$$n(n+1)+1$$

Local min = Global min



	= ?	#neigh-bors	poly-time algorithm local opt	global opt
submodular (set fn)	Y	2^n		
separable-conv	Y	$2n$		
integrally-conv	Y	3^n		
$\mathrm{L}^\natural$-conv $(\mathbb{Z}^n)$	Y	2^n		
$\mathrm{M}^\natural$-conv $(\mathbb{Z}^n)$	Y	n^2		

Local min = Global min



	$=$?	#neigh -bors	poly-time algorithm local opt	poly-time algorithm global opt
submodular (set fn)	Y	2^n	Y	
separable-conv	Y	$2n$	Y	
integrally-conv	Y	3^n	N	
L$^\natural$-conv ($\mathbb{Z}^n$)	Y	2^n	Y	
M$^\natural$-conv ($\mathbb{Z}^n$)	Y	n^2	Y	

P3.

Operations

Operations

- scaling: $af(x) + b$, $\quad f(ax + b)$
- linear addition: $f(x) + \langle p, x \rangle$
- section: $f(x, 0)$
- projection: $\min_y f(x, y)$

- sum: $f_1(x) + f_2(x)$
- convolution: $(f_1 \Box f_2)(x) = \min_y (f_1(y) + f_2(x - y))$

- transformation by graphs/networks

Sum and Convolution

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

Theorem: (Murota 98)

f_1, f_2 : L $\Longrightarrow$ $f_1 + f_2$: L

L$^\natural$ $\Longrightarrow$ L$^\natural$

$f_1, f_2, \ldots, f_k$: L (L$^\natural$) $\Longrightarrow$ $f_1 + f_2 + \cdots + f_k$: L (L$^\natural$)

- $(f_1 \Box f_2)(x) = \min_y (f_1(y) + f_2(x - y))$

Theorem: (Murota 96)

f_1, f_2 : M $\Longrightarrow$ $f_1 \Box f_2$: M

M$^\natural$ $\Longrightarrow$ M$^\natural$

$f_1, f_2, \ldots, f_k$: M (M$^\natural$) $\Longrightarrow$ $f_1 \Box f_2 \Box \cdots \Box f_k$: M (M$^\natural$)

Rem: M $+$ M is not M, L $\Box$ L is not L

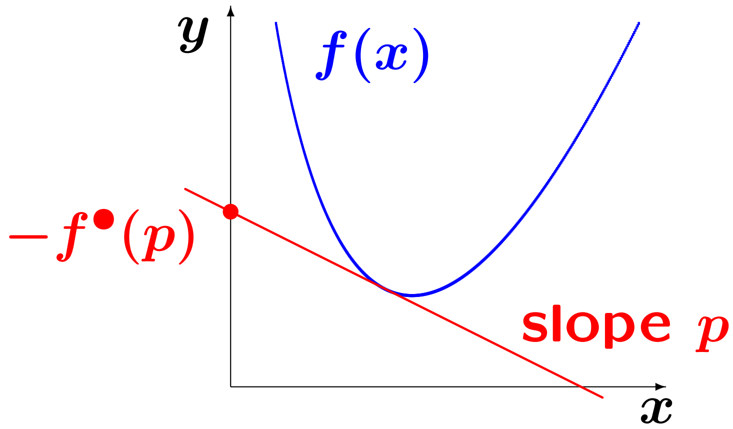
Sum/Convolution

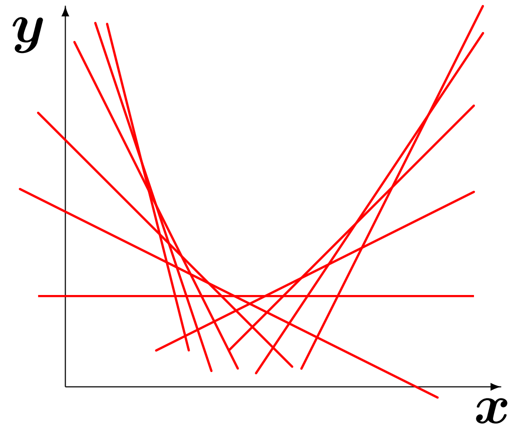
	sum $f_1 + f_2$	convolution $f_1 \square f_2$
submodular (set fn)	Y	N matroid intersec $\min_{Y \subseteq X} (\rho_1(Y) + \rho_2(X \setminus Y))$
separable-conv	Y	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	Y	N $\to$ L$_2$-convex
M-conv ($\mathbb{Z}^n$)	N $\to$ M$_2$-conv matr.intersec	Y matroid union

P4.
Conjugacy

(Legendre transform)

Conjugacy: Discrete Legendre Transform





$$f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} \{\langle p, x \rangle - f(x)\}$$

$\Rightarrow$ **If** $f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$, **then** $f^{\bullet} : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

(integer-valued)

M-L Conjugacy Theorem

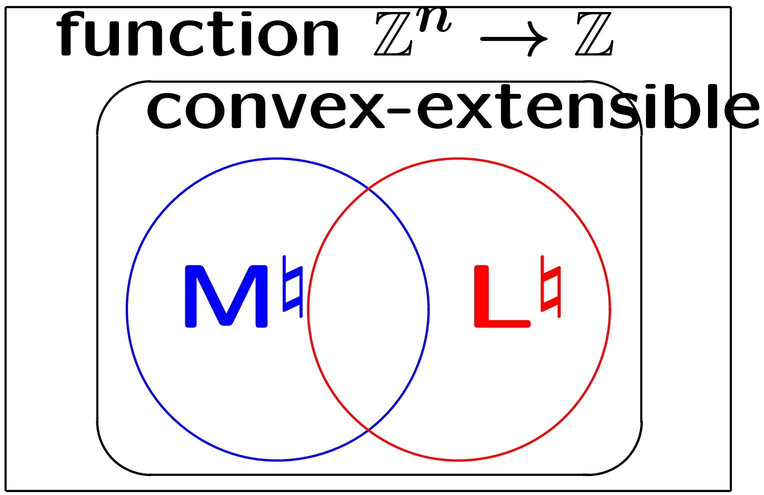
Integer-valued discrete fn $\quad f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$

Legendre transform: $\quad f^{\bullet}(p) = \sup_{x \in \mathbb{Z}^n} [\langle p, x \rangle - f(x)]$

(1) **M and L are conjugate** (Murota 98)

(2) **M♮ and L♮ are conjugate**

$$f \mapsto f^{\bullet} = g \mapsto g^{\bullet} = f$$



(3) **biconjugacy**

$$f^{\bullet\bullet} = f$$

for $f \in$ M♮ $\cup$ L♮

Conjugacy and Biconjugacy

$f : \mathbb{Z}^n \to \overline{\mathbb{Z}}$	$f^\bullet : \mathbb{Z}^n \to \overline{\mathbb{Z}}$	$f^{\bullet\bullet} = f$
submodular (set fn)	submodular polyhedron $\{x \in \mathbb{Z}^n \mid x(A) \leq \rho(A)\}$	Y
separable-convex $f(x) = \sum \varphi_i(x_i)$	separable-convex $\varphi_1^\bullet(p_1) + \cdots + \varphi_n^\bullet(p_n)$	Y
integrally-convex	N	N
L-convex $(\mathbb{Z}^n)$	M-convex	Y
L$^\natural$-convex	M$^\natural$-convex	Y
M-convex $(\mathbb{Z}^n)$	L-convex	Y
M$^\natural$-convex	L$^\natural$-convex	Y

Significance of M-L Conjugacy

- Economics (game, auction)

  x: commodity bundle, p: price vector

- Network flow (min-cost flow)

  x: flow, p: tension (potential)

- Electrical network (Iri's book 69)

  x: current, p: voltage (potential)

- Discrete DC programming (Maehara-Murota 15)

- Convexity View on Matroids

Conjugacy in Linear Algebra

$$[a_1, \cdots, a_5] = \begin{array}{|c|c|c|c|c|} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

Bases $\mathcal{B} = \{ \{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,4,5\}, \{3,4,5\} \}$

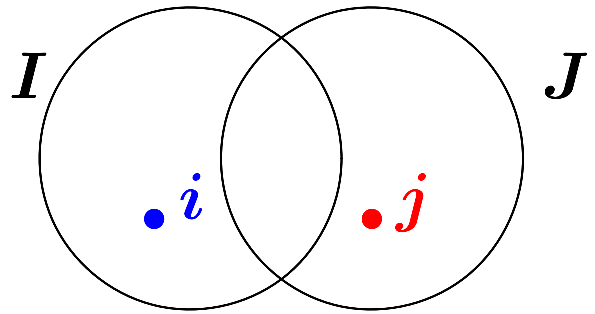
Rank fn $\rho(X) = \text{rank}\,\{a_j \mid j \in X\}$

Equivalence $\mathcal{B} \iff \rho$

$$\rho(X) = \max\{|X \cap J| \mid J \in \mathcal{B}\} \qquad (X \subseteq V)$$

$$\mathcal{B} = \{J \subseteq V \mid \rho(J) = |J| = \rho(V)\}$$

Axioms of Matroid

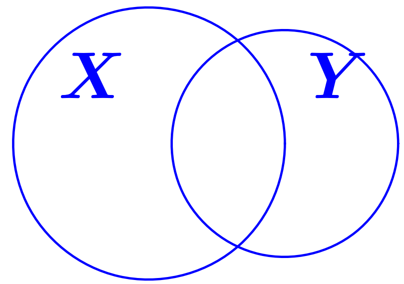


Basis axiom (set family $\mathcal{B}$):

$\forall\ I, J \in \mathcal{B},\ i \in I \setminus J, \quad \exists\ j \in J \setminus I$:

$$I - i + j \in \mathcal{B},\ J + i - j \in \mathcal{B}$$

Rank axiom (set function ρ):



(R1) $0 \leq \rho(X) \leq |X|$

(R2) $X \subseteq Y \implies \rho(X) \leq \rho(Y)$

(R3) $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$

Equivalence $\mathcal{B} \Leftrightarrow \rho$ (M $\leftrightarrow$ L by Legendre tr.)

$$\rho(X) = \max\{|X \cap J| \mid J \in \mathcal{B}\} \qquad (X \subseteq V)$$

$$\mathcal{B} = \{J \subseteq V \mid \rho(J) = |J| = \rho(V)\}$$

Dual Character of Matroid Rank

$$\rho(X) = \max\{|I| \mid I : \text{independent},\ I \subseteq X\}$$

is M$^\natural$-convex and L$^\natural$-concave

Edmonds' matroid union formula:

$$\max_X\{\rho_1(X) + \rho_2(V \setminus X)\} = \min_Y\{\rho_1(Y) + \rho_2(Y) + |V \setminus Y|\}$$

submod maximization (M$^\natural$-concave $\Box$ M$^\natural$-concave)

submod minimization (L$^\natural$-convex + L$^\natural$-convex)

Self-Conjugacy: $\rho(X) = |X| - \rho^\bullet(\chi_X)$

History of Discrete Conjugacy

Matroid bases Whitney 35	$\longleftrightarrow$	Matroid rank fn Whitney 35
$\Downarrow$		$\Downarrow$
Polymatroid Edmonds 70	$\longleftrightarrow$	**Submodular fn** Edmonds 70
$\Downarrow$		$\Downarrow$
Valuated matroid Dress–Wenzel 90	|	Lovász extension Lovász 83
$\Downarrow$	|	$\Downarrow$
	|	Submod. integ. conv. fn Favati-Tardella 90
		$\Downarrow$
M-convex fn Murota 96	$\longleftrightarrow$	**L-convex fn** Murota 98
$\Updownarrow$		$\Updownarrow$
$M^\natural$-convex fn Murota -Shioura 99	$\longleftrightarrow$	$L^\natural$-convex fn Fujishige–Murota 00

P5.
Duality

(separation theorem)
(Fenchel duality)

Matroid Intersection Problem

Given two matroids ......

- Find a common indep. set X with max $|X|$
- Find a common base B (if any)

Given two matroids and weight w ......

- Find a common indep. set X with max $w(X)$
- Find a common base B with max $w(B)$

LP formulation with integrality, combinatorial duality, efficient algorithms, many applications

Edmonds' Intersection Theorem

Submodular polyhedron $(\rho(\emptyset) = 0,\ \rho(V) < +\infty)$

$$\mathrm{P}(\rho) = \{x \in \mathbb{R}^n \mid x(X) \leq \rho(X)\ (\forall X \subseteq V)\} \qquad (|V| = n)$$

Theorem: (Edmonds 70)

(1) For $\rho_1, \rho_2 : 2^V \to \overline{\mathbb{R}}$: submodular,

$$\max_x \{x(V) \mid x \in \mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2)\} = \min_X \{\rho_1(X) + \rho_2(V \setminus X)\}$$

(2) If ρ_1 and ρ_2 are integer-valued, then

$$\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) = \overline{\mathrm{P}(\rho_1) \cap \mathrm{P}(\rho_2) \cap \mathbb{Z}^n}$$

and there exists $x^* \in \mathbb{Z}^n$ that attains the maximum

Frank's Discrete Separation

(Frank 82)

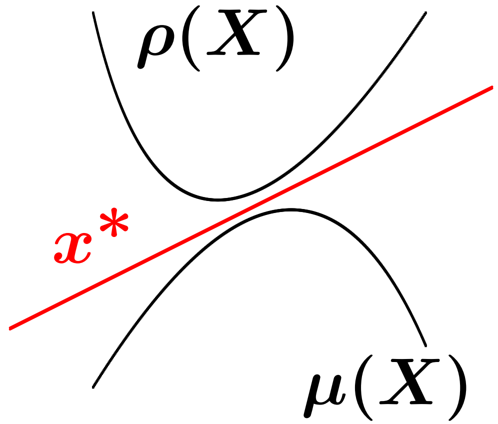
$\rho : 2^V \to \mathbb{R}$: **submodular** $\quad (\rho(\emptyset) = 0)$

$\mu : 2^V \to \mathbb{R}$: **supermodular** $\quad (\mu(\emptyset) = 0)$

- $\rho(X) \geq \mu(X) \quad (\forall X \subseteq V) \Rightarrow \; \exists \; x^* \in \mathbb{R}^V$:

$$\rho(X) \geq x^*(X) \geq \mu(X) \qquad (\forall X \subseteq V)$$

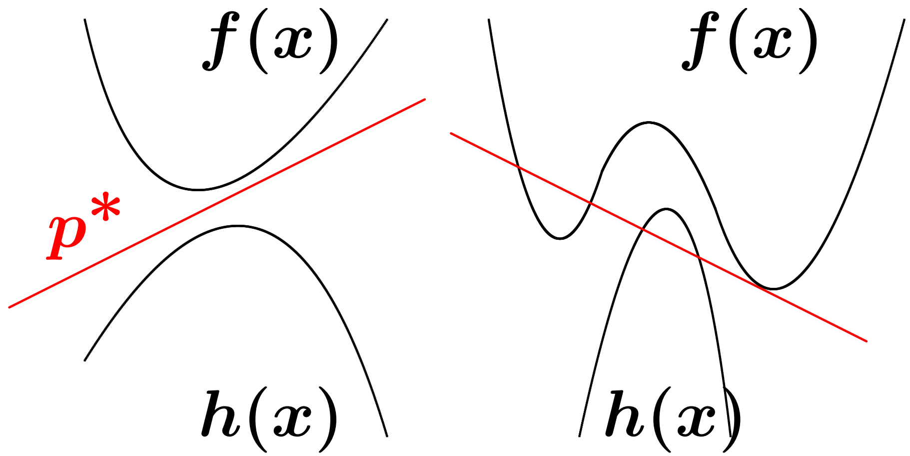
- ρ, μ: **integer-valued** $\Rightarrow x^* \in \mathbb{Z}^V$



Discrete Separation Theorem

$f : \mathbb{Z}^n \to \mathbb{R}$ "convex"

$h : \mathbb{Z}^n \to \mathbb{R}$ "concave"



- $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}^n) \Rightarrow \exists\ \alpha^* \in \mathbb{R},\quad \exists\ p^* \in \mathbb{R}^n$:

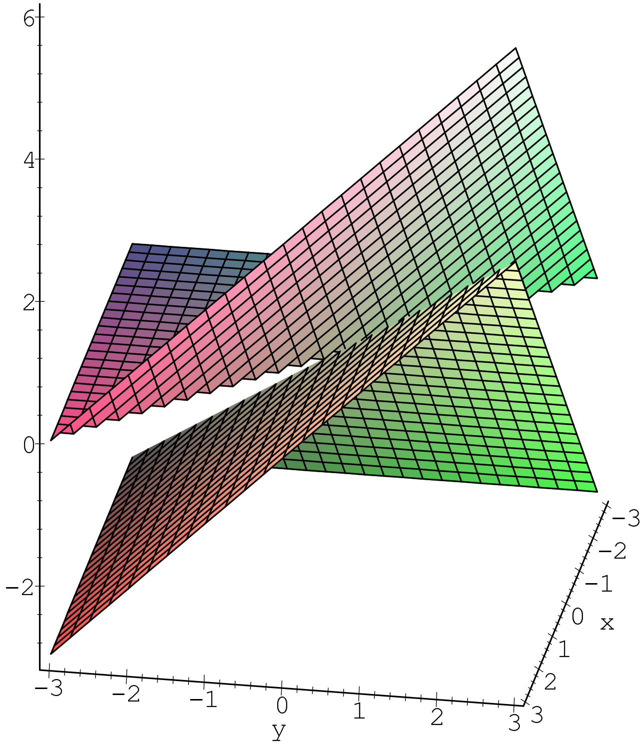
$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x) \qquad (x \in \mathbb{Z}^n)$$

- f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z},\quad p^* \in \mathbb{Z}^n$

Difficulty of Discrete Separation (1)

$f(x, y) = \max(0, x + y)$ **convex**

$h(x, y) = \min(x, y)$ **concave**



nonintegral separation

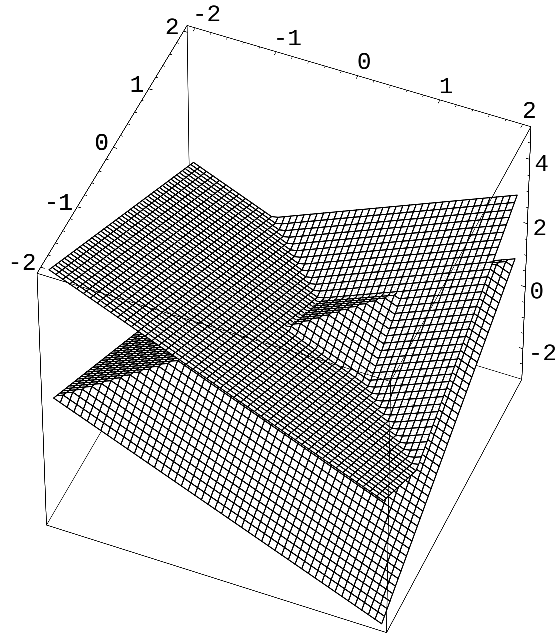
$p^* = (1/2, 1/2)$, $\alpha^* = 0$ **unique separating plane**

Difficulty of Discrete Separation (2)

Even real-separation is nontrivial

$f(x,y) = |x+y-1|$ **convex**

$h(x,y) = 1 - |x-y|$ **concave**



- $f(x,y) \geq h(x,y)$ $(\forall (x,y) \in \mathbb{Z}^2)$ **true**
- **No** $\alpha^* \in \mathbb{R}$, $p^* \in \mathbb{R}^2$: $f(x) \geq \alpha^* + \langle p^*, x \rangle \geq h(x)$

$\because$ $f = 0 < h = 1$ **at** $(x,y) = (1/2, 1/2)$

Discrete Separation Theorems

(Murota 96/98)

M-separation Thm (for M$^\natural$-convex)

$\Rightarrow$ Weight splitting for weighted matroid intersection

(Iri-Tomizawa 76, Frank 81)

(linear fn, indicator fn = M$^\natural$-convex fn)

L-separation Thm (for L$^\natural$-convex)

$\Rightarrow$ Discrete separ. for submod. set fn (Frank 82)

(submod. set fn = L$^\natural$-convex fn on 0–1 vectors)

Min-Max Duality

f: M$^\natural$-convex, h: M$^\natural$-concave $(\mathbb{Z}^n \to \overline{\mathbb{Z}})$

Legendre–Fenchel transform

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbb{Z}^n\}$$

$$h^\circ(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbb{Z}^n\}$$

Fenchel-type duality thm (Murota 96, 98)

$$\inf_{x\in\mathbb{Z}^n}\{f(x) - h(x)\} = \sup_{p\in\mathbb{Z}^n}\{h^\circ(p) - f^\bullet(p)\}$$

self-conjugate ($f^\bullet$: L$^\natural$-convex, h°: L$^\natural$-concave)

$\Longrightarrow$ Edmonds' intersection thm
Fujishige's Fenchel duality thm

Relation among Duality Thms

Discrete Convex	Combinatorial Opt.
M-separation $f(x) \geq \boxed{\text{Lin}} \geq h(x)$	Fenchel duality (Fujishige 84) matroid intersect. (Edmonds 70)
$\Updownarrow$	$\Updownarrow$
Fenchel duality $\inf\{f - h\} = \sup\{h^{\circ} - f^{\bullet}\}$	$\Rightarrow$ discrete separ. for submod (Frank 82) $\Rightarrow$ valuated matroid intersect. (M. 96)
$\Updownarrow$	$\Downarrow$
L-separation $f^{\bullet}(p) \geq \boxed{\text{Lin}} \geq h^{\circ}(p)$	weighted matroid intersect. (Edmonds 79, Iri-Tomizawa 76, Frank 81)

Separation and Min-Max Theorems

	separation	min-max
submodular (set fn)	Y (Frank)	Y (Edmonds, Fujishige)
separable-conv	Y	Y
integrally-conv	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y
M-conv ($\mathbb{Z}^n$)	Y	Y

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca ling	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	Y	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	–	–	Y	Y*	Y*	?	?	?

Summary

	Operations				Minimize		Conjugacy/Duality			
	sca lng	sum	cnvl tion	graf tran	loc glob	prox mity	cnv ext	bi- cnj	sep thm	min max
submod (set fn)	–	Y	N	Y*	Y	–	Y	Y	Y	Y
separ -conv	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
integ -conv	Y*	N	N	N	Y	Y*	Y	N	N	N
L-conv ($\mathbb{Z}^n$)	Y	Y	N	Y*	Y	Y	Y	Y	Y	Y
M-conv ($\mathbb{Z}^n$)	N	N	Y	Y	Y	Y	Y	Y	Y	Y
M-conv (jump)	N	N	Y	Y	Y	?	N	N	N	N
L-conv (tree^n)	?	Y	–	–	Y	Y*	Y*	?	?	?

Five Properties of "Convex" Functions

1. convex extension
2. local opt $=$ global opt
3. Conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for

- separable-convex functions
- L$^\natural$-convex functions
- M$^\natural$-convex functions

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