ICCOPT 2016, Summer School, Tokyo, August 7, 2016

Convex Analysis Approach to Discrete Optimization, I

Concepts of Discrete Convex Functions

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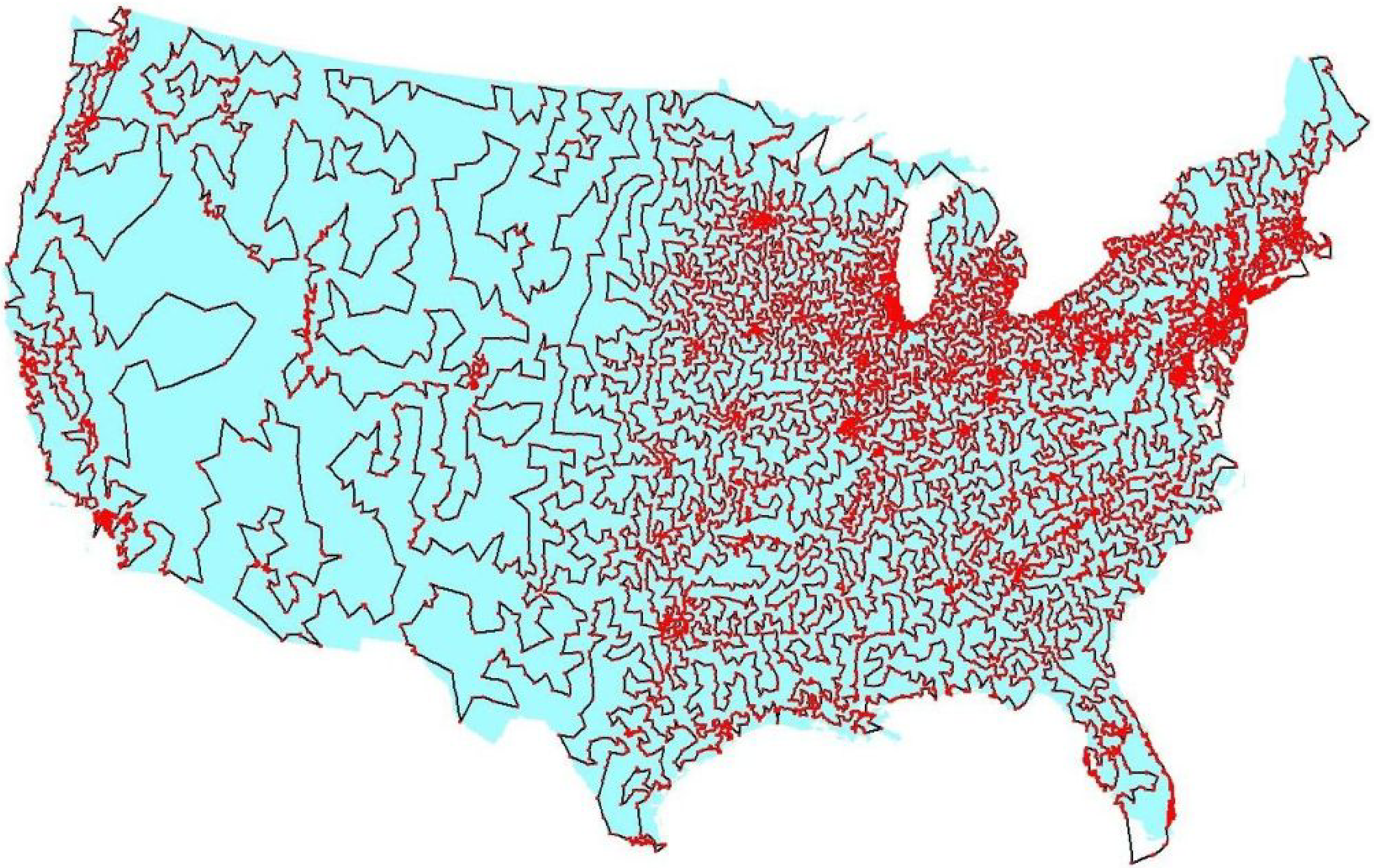
160807ICCOPT11

Discrete Optimization

Traveling Salesman Problem
Minimum Spanning Tree Problem

Traveling Salesman Problem

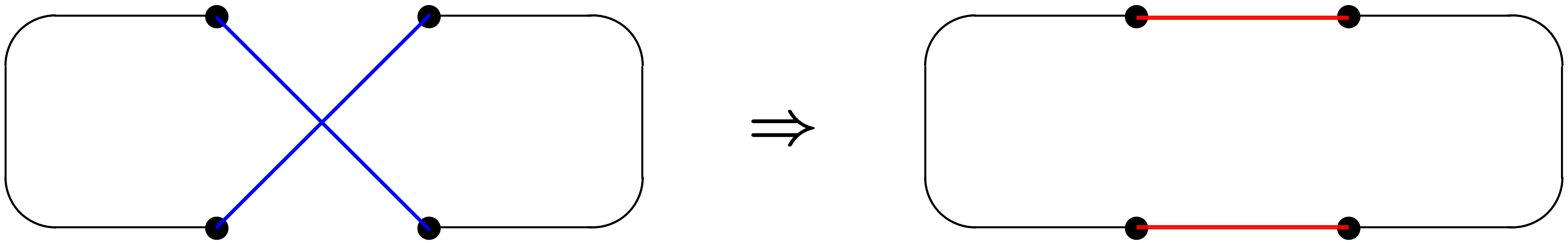


13,509 cities, D. Applegate, R. Bixby, V. Chvatal, and W. Cook (1998)

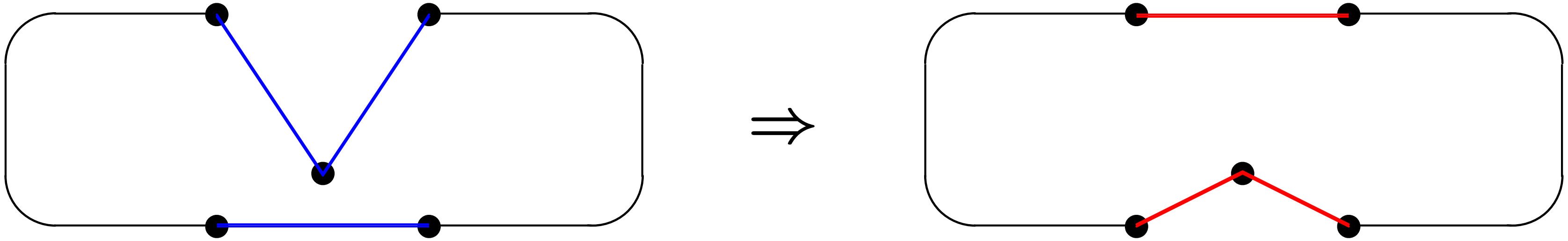
http://www.tsp.gatech.edu/history/pictorial/usa13509.html

Local Improvements in TSP

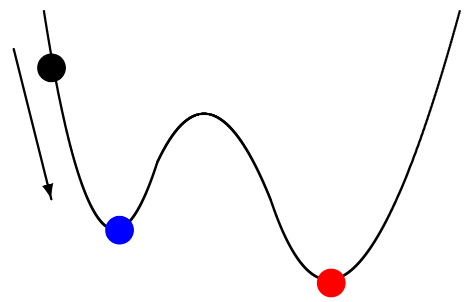
2-opt



Or-opt



Local Search (Descent Method)



S0: Initial sol x^*

S1: Minimize $f(x)$ in nbhd of x^* to obtain $x^\bullet$

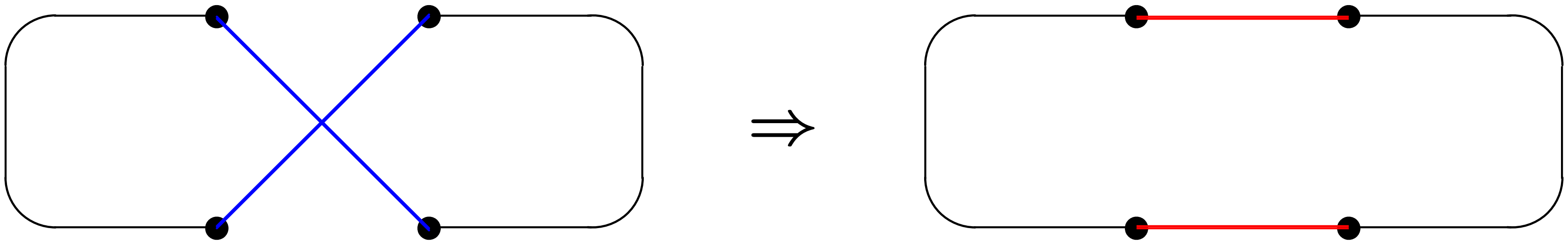
S2: If $f(x^*) \leq f(x^\bullet)$, return x^* (local opt)

S3: Update $x^* := x^\bullet$; go to S1

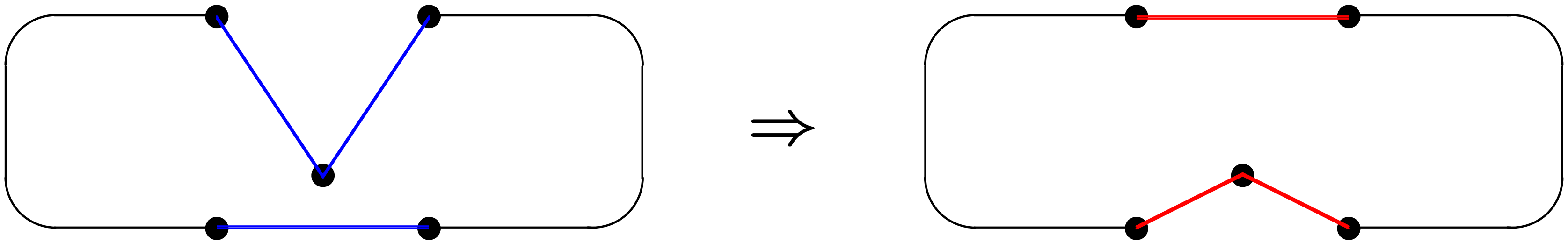
What is neighborhood?

Neighborhood in TSP

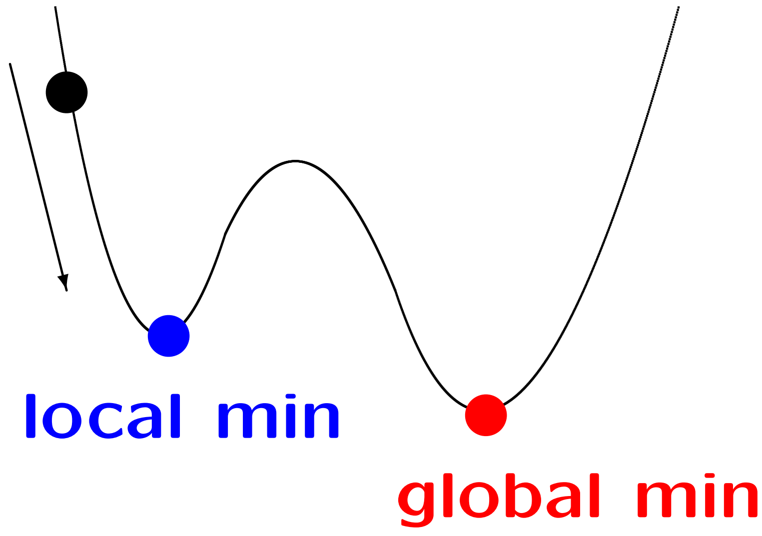
2-opt



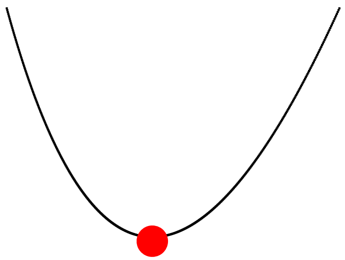
Or-opt



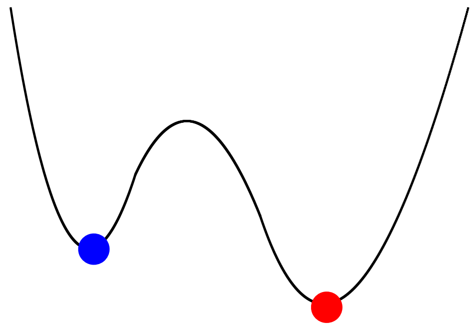
Descent Method and Convexity



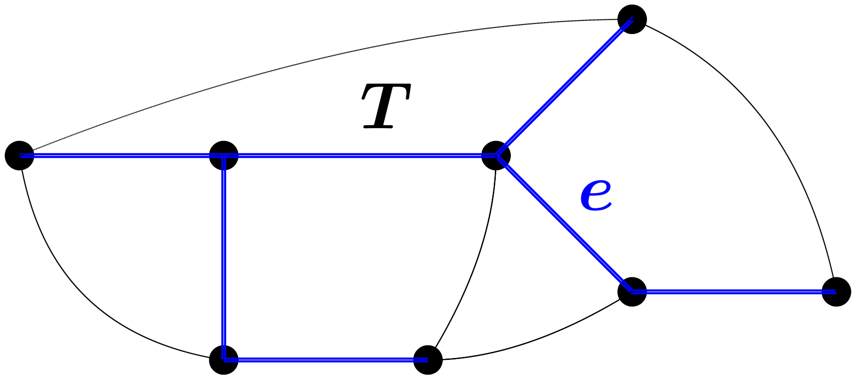
convex



nonconvex



Min Spanning Tree Problem

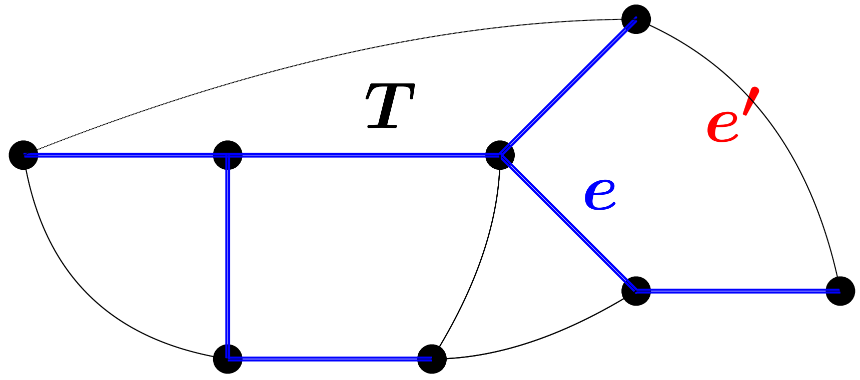


edge length $d : E \to \mathbb{R}$

total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Min Spanning Tree Problem



edge length $d : E \to \mathbb{R}$

total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Thm

T: MST $\iff \tilde{d}(T) \le \tilde{d}(T - e + e')$ (local min)

$\iff d(e) \le d(e')$ if $T - e + e'$ is tree

Algorithms Kruskal, Kalaba;
Prim, Dijkstra, Borůvka, Jarnik, $\cdots$

Kruskal's Greedy Algorithm for MST

Kruskal (1959)

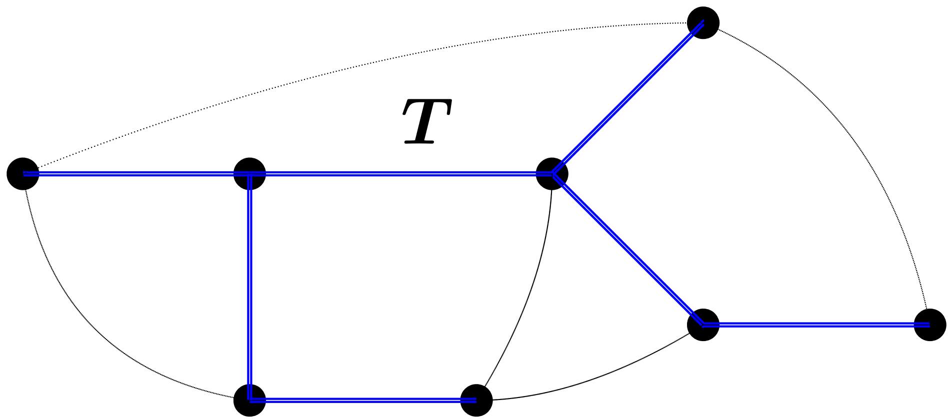
S0: Order edges by length: $d(e_1) \le d(e_2) \le \cdots$

S1: $T = \emptyset$; $i = 1$

S2: Pick edge e_i

S3: If $T + e_i$ contains a cycle, discard e_i

S4: Update $T = T + e_i$; $i = i + 1$; go to S2



Kalaba's Algorithm for MST

Kalaba (1960), Dijkstra (1960)

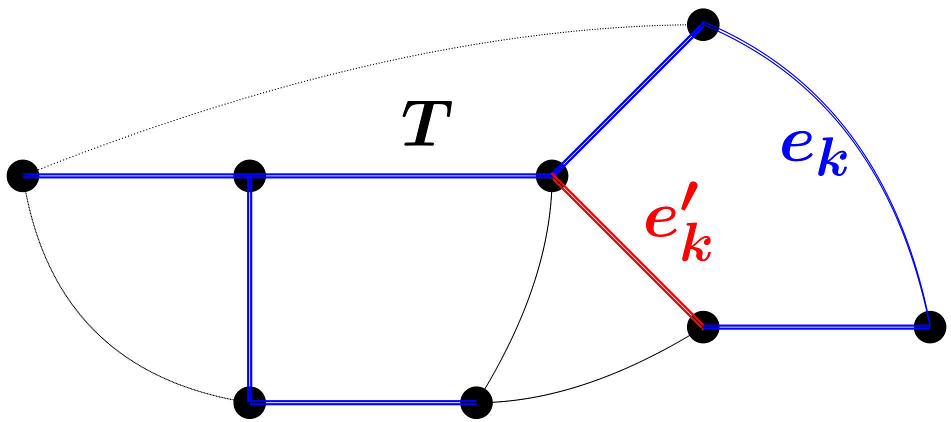
S0: $T =$ any spannning tree

S1: Order $e' \notin T$ by length: $d(e'_1) \leq d(e'_2) \leq \cdots$

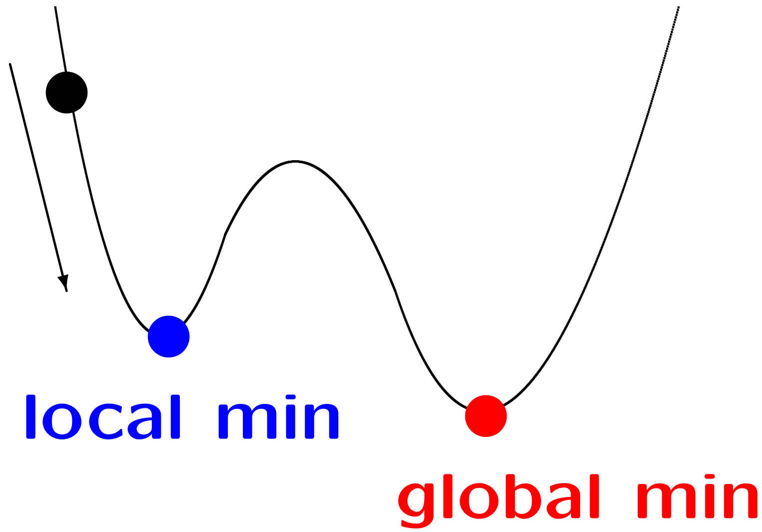
$k = 1$

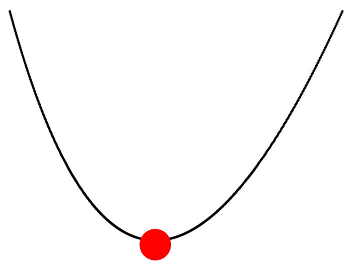
S2: $e_k =$ longest edge s.t. $T - e_k + e'_k$ is tree

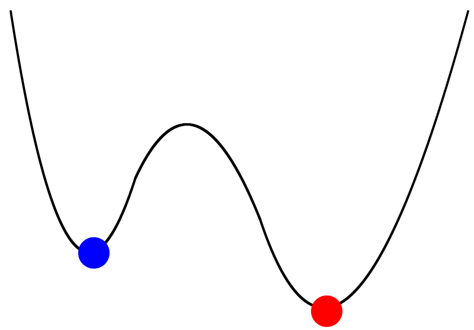
S3: $T = T - e_k + e'_k$; $k = k + 1$; go to S2



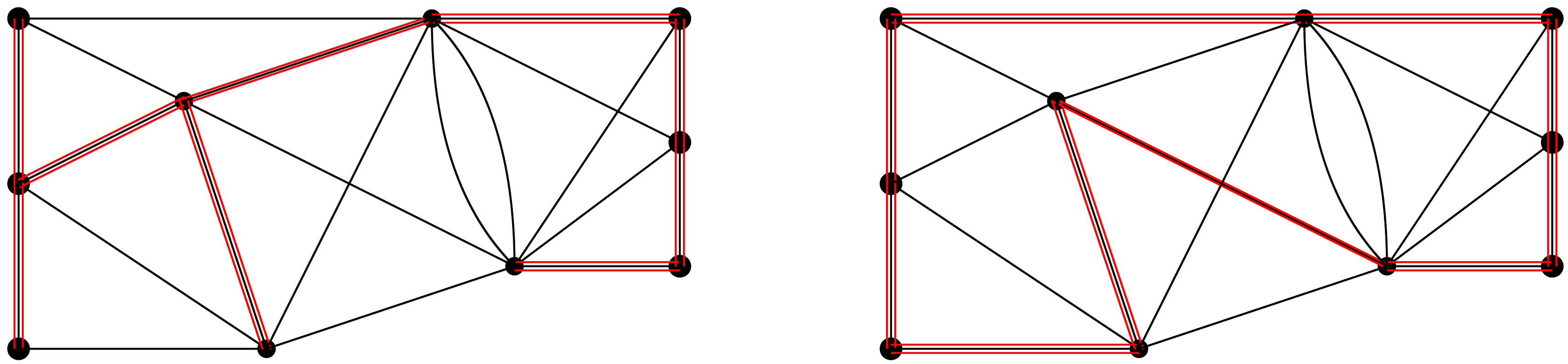
Descent Method and Convexity







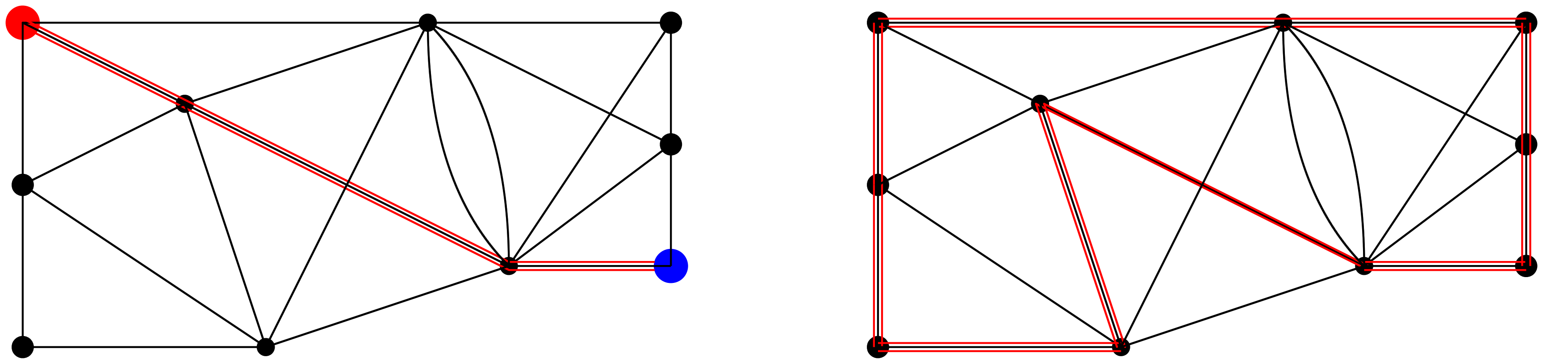
Min.Span.Tree vs Traveling Salesman



	Min.Span.Tree	Travel.Salesman
Empirical Fact	easy	difficult
Complexity Theory	poly-time	NP-hard (?)
Convexity View	"convex"	non-"convex"

- Kruskal/Kalaba for MST
  = steepest descent for M-convex minimization

Shortest Path vs Traveling Salesman



	Shortest Path	Travel.Salesman
Empirical Fact	easy	difficult
Complexity Theory	poly-time	NP-hard (?)
Convexity View	"convex"	non-"convex"

- Dijkstra for shortest path
  = steepest descent for L-convex minimization

Contents of Part I

Concepts of Discrete Convex Functions

C1. Univariate Discrete Convex Functions

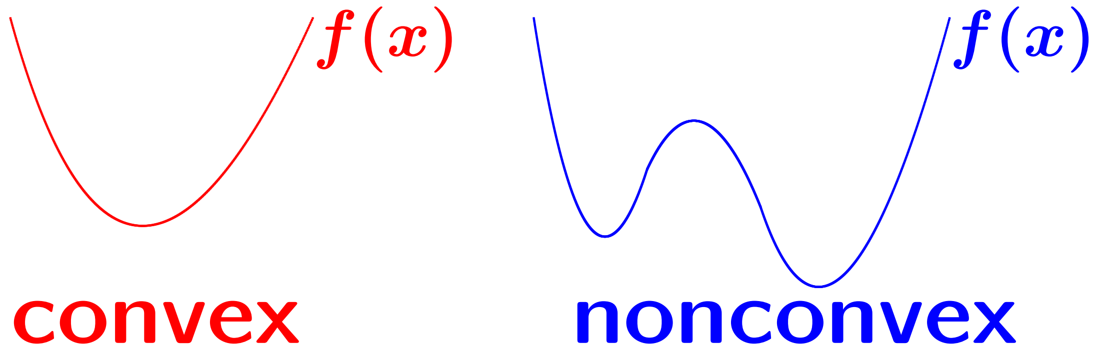
C2. Classes of Discrete Convex Functions

C3. L-convex Functions

C4. M-convex Functions

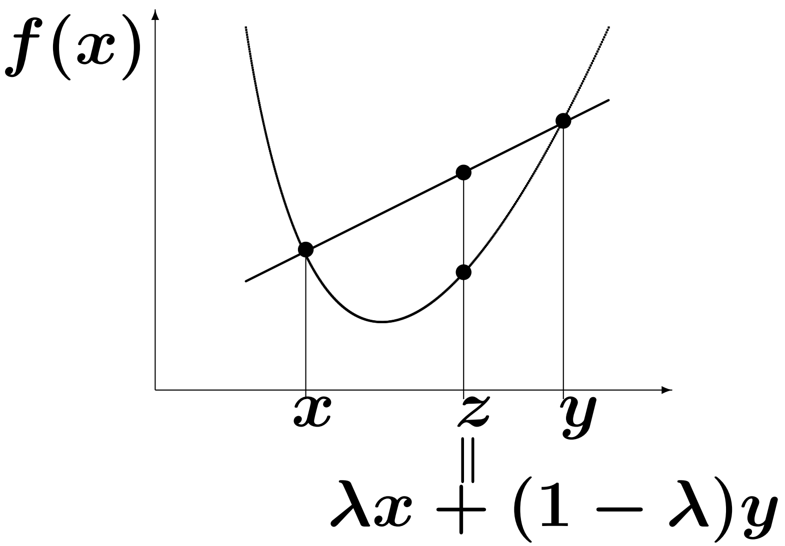
Part II: Properties, Part III: Algorithms

Convex Function



$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex $\iff$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \qquad (0 < \forall\ \lambda < 1)$$



$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$$

C1.
Univariate
Discrete Convex Functions

(Ingredients of convex analysis)

Definition of "Convex" Function

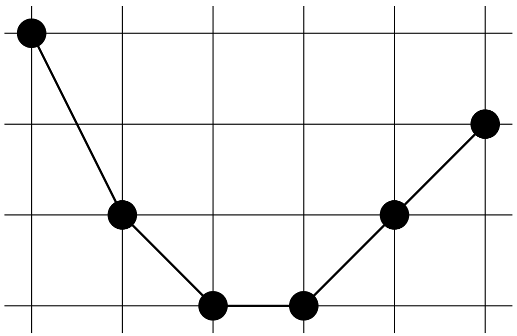
$f : \mathbb{Z} \to \overline{\mathbb{R}}$ $\qquad\qquad$ $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

$$f(x-1) + f(x+1) \geq 2f(x)$$

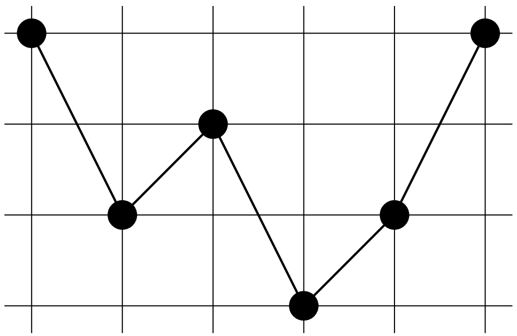
$$\iff \quad f(x) + f(y) \geq f(x+1) + f(y-1) \qquad (x < y)$$

$\iff$ f **is convex-extensible, i.e.,**

$\exists$ **convex** $\overline{f} : \mathbb{R} \to \overline{\mathbb{R}}$ **s.t.** $\overline{f}(x) = f(x)$ $(\forall x \in \mathbb{Z})$



convex



non-convex

Local vs Global Optimality

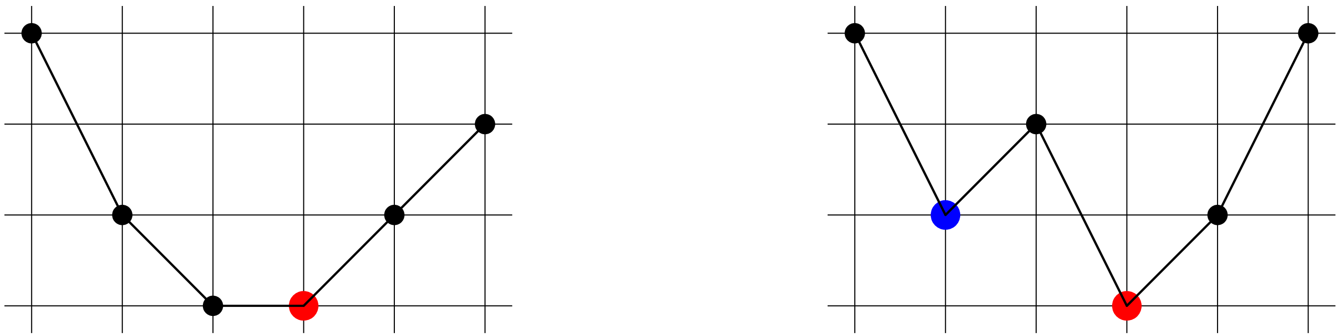
$f : \mathbb{Z} \to \overline{\mathbb{R}}$

Theorem:

x^*: global opt (min)

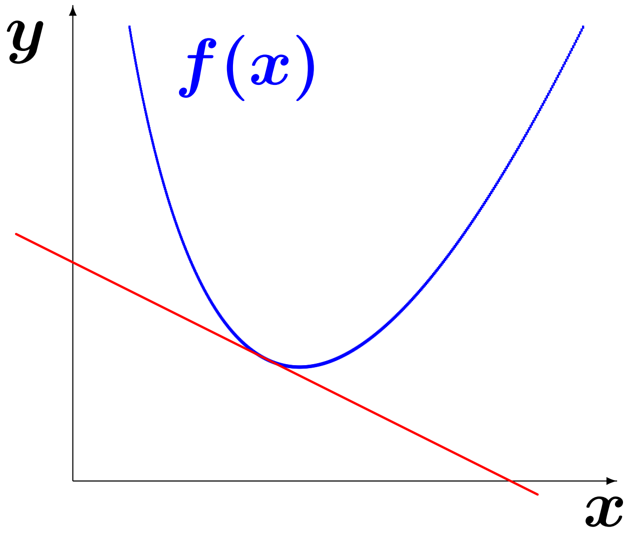
$\iff$ x^*: local opt (min)

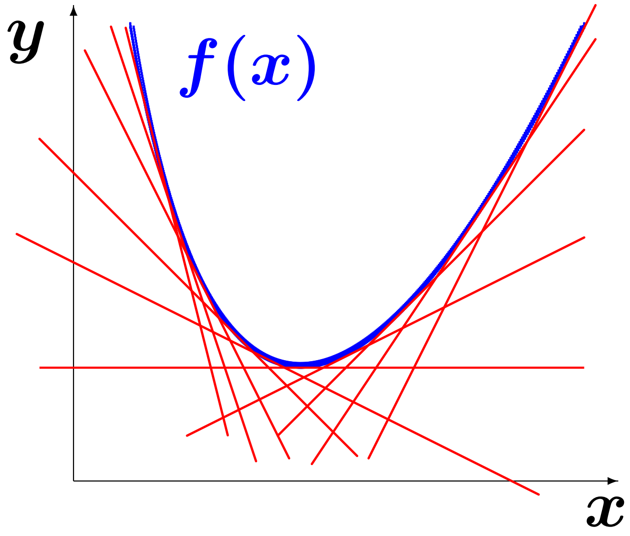
$$f(x^*) \leq \min\{f(x^* - 1),\ f(x^* + 1)\}$$



convex　　　　non-convex

Legendre Transformation (Intuition)





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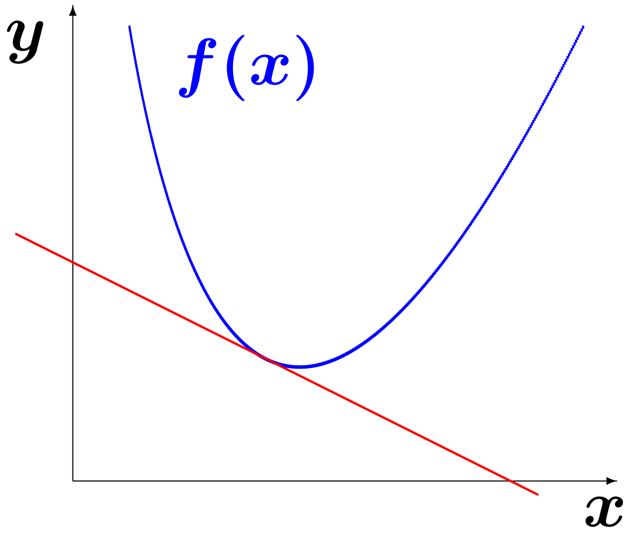

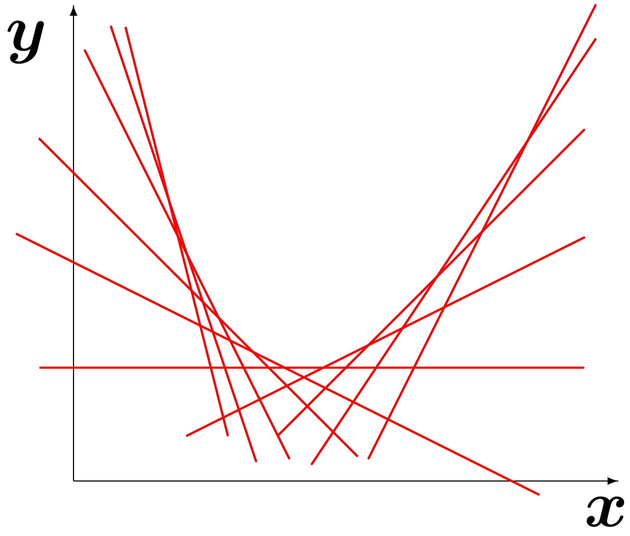

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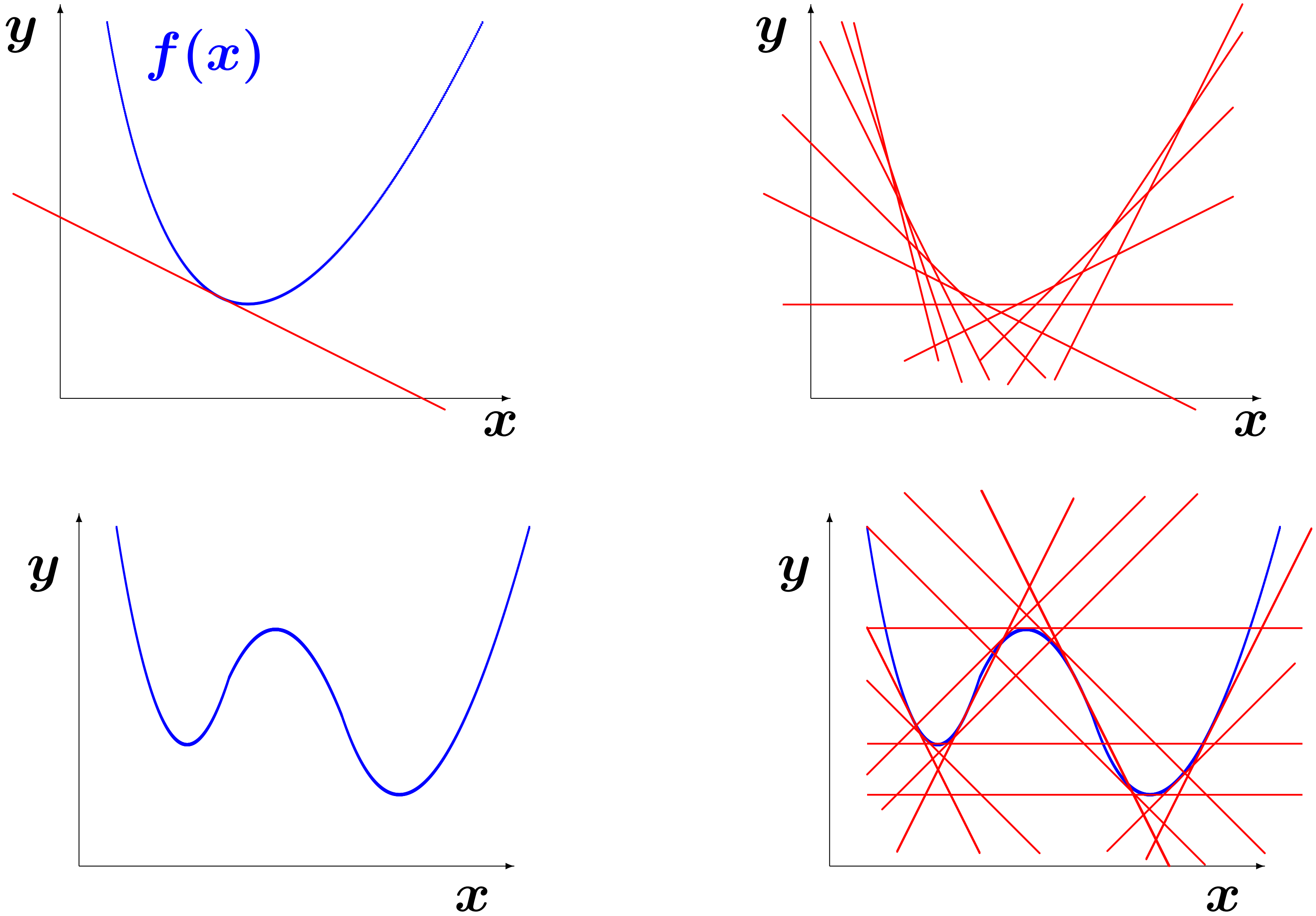
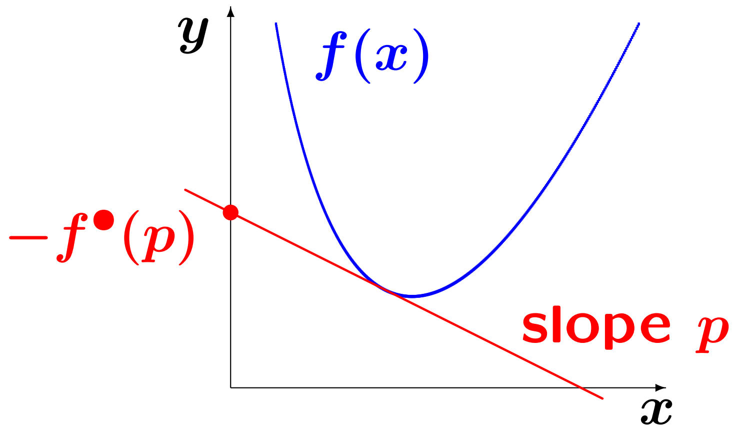


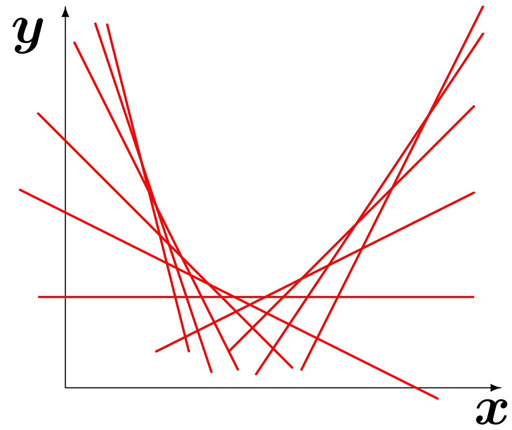
Legendre Transformation

$f : \mathbb{Z} \to \overline{\mathbb{Z}}$ (integer-valued)

Define discrete Legendre transform of f by

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\} \qquad (p \in \mathbb{Z})$$





Theorem:

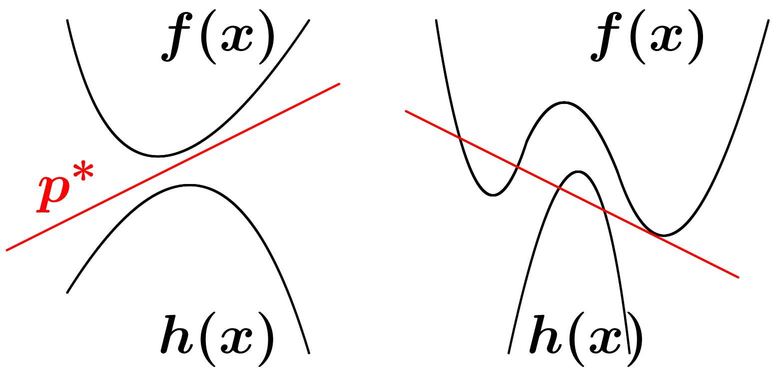
(1) $f^{\bullet}$ is $\mathbb{Z}$-valued convex function, $f^{\bullet} : \mathbb{Z} \to \overline{\mathbb{Z}}$

(2) $(f^{\bullet})^{\bullet} = f$ (biconjugacy)

Separation Theorem

$f : \mathbb{Z} \to \overline{\mathbb{R}}$
convex

$h : \mathbb{Z} \to \underline{\mathbb{R}}$
concave



Theorem (Discrete Separation Theorem)

(1) $f(x) \geq h(x) \quad (\forall x \in \mathbb{Z}) \Rightarrow \; \exists\, \alpha^* \in \mathbb{R}, \; \exists\, p^* \in \mathbb{R}$:

$$f(x) \geq \alpha^* + p^* x \geq h(x) \quad (\forall x \in \mathbb{Z})$$

(2) f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z}, \; p^* \in \mathbb{Z}$

Separation Theorem

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convex

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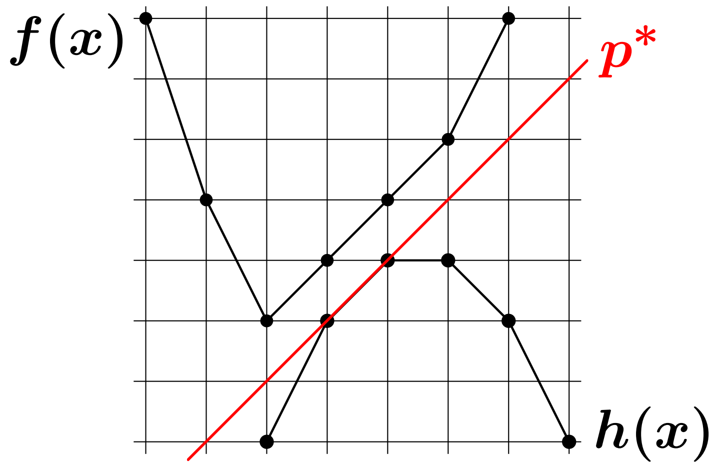

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(2) f, h: **integer-valued** $\Rightarrow \alpha^* \in \mathbb{Z}, \ \ p^* \in \mathbb{Z}$

Fenchel Duality (Min-Max)

$f : \mathbb{Z} \to \overline{\mathbb{Z}}$: convex, $\qquad h : \mathbb{Z} \to \underline{\mathbb{Z}}$: concave

Legendre transforms:

$$f^{\bullet}(p) = \sup\{px - f(x) \mid x \in \mathbb{Z}\}$$

$$h^{\circ}(p) = \inf\{px - h(x) \mid x \in \mathbb{Z}\}$$

Theorem:

$$\inf_{x \in \mathbb{Z}} \{f(x) - h(x)\} = \sup_{p \in \mathbb{Z}} \{h^{\circ}(p) - f^{\bullet}(p)\}$$

Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **univariate**
discrete convex functions

C2.

Classes of

Discrete Convex Functions

Classes of Discrete Convex Functions

1. Submodular set fn (on $\{0,1\}^n$)
1. Separable-convex fn on $\mathbb{Z}^n$
1. Integrally-convex fn on $\mathbb{Z}^n$

2. L-convex (L$^\natural$-convex) fn on $\mathbb{Z}^n$
2. M-convex (M$^\natural$-convex) fn on $\mathbb{Z}^n$

3. M-convex fn on jump systems
3. L-convex fn on graphs

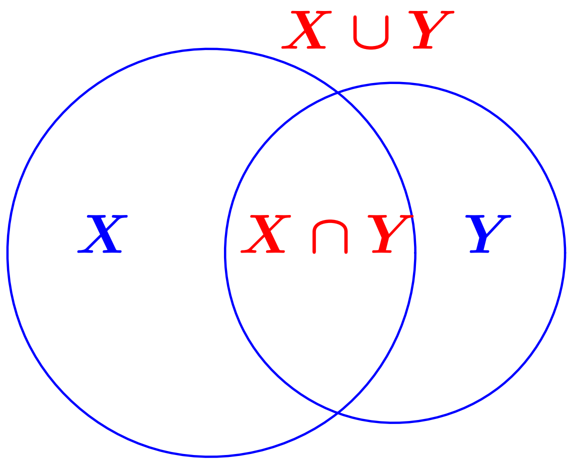
Submodular Function

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

Set function $\rho : 2^V \to \overline{\mathbb{R}}$ is submodular

$\iff$

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$



cf. $|X| + |Y| = |X \cup Y| + |X \cap Y|$

Set function $\iff$ Function on $\{0,1\}^n$

Submodularity & Convexity in 1980's

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

- min/max algorithms

  (Grötschel–Lovász–Schrijver/ Jensen–Korte, Lovász)

  min $\Rightarrow$ polynomial, max $\Rightarrow$ exponential

- Convex extension (Lovász)

  set fn is submod $\Leftrightarrow$ Lovász ext is convex

- Duality theorems (Edmonds, Frank, Fujishige)

  discrete separation, Fenchel min-max

Submodular set functions
= Convexity + Discreteness

Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **submodular set functions**

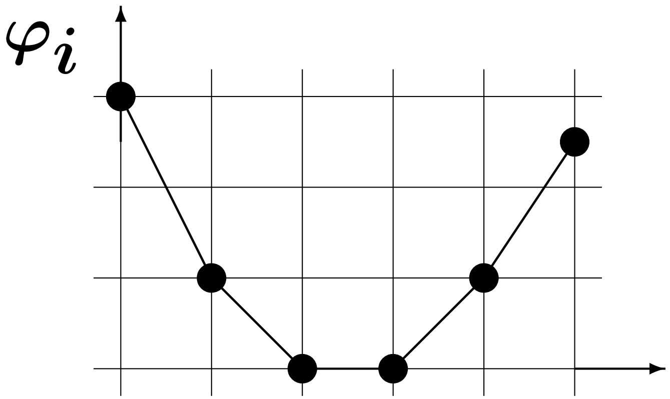
Separable-convex Function

$f : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is **separable-convex**

$\iff$

$$f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)$$

φ_i: univariate convex



Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **separable discrete convex functions**

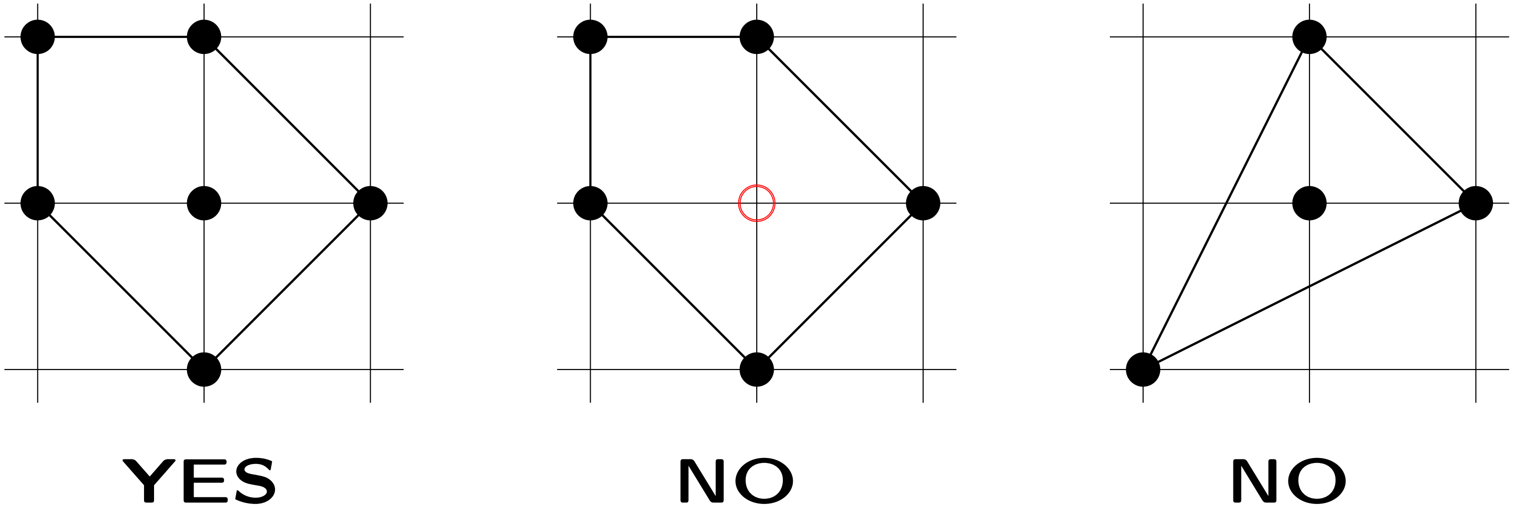
Some History

1935	Matroid	Whitney, Nakasawa
1965	Submodular function	Edmonds
1969	Convex network flow (electr.circuit)	Iri
1982	Submodularity and convexity	Frank, Fujishige, Lovász
1990	Valuated matroid	Dress–Wenzel
	Integrally convex fn	Favati–Tardella
1996	Discrete convex analysis	Murota
2000	Submodular minimization algorithm	Iwata–Fleischer–Fujishige, Schrijver
2006	M-convex fn on jump system	Murota
2012	L-convex fn on graph	Hirai, Kolmogorov

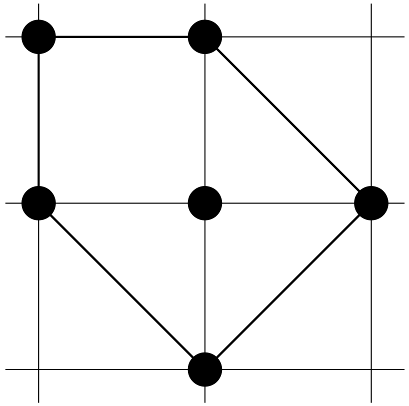
Motivations/Applications/Connections

1. submodular	MANY problems graph cut, convex game
1. separable-conv	MANY problems min-cost flow, resource allocation
1. integrally-conv	economics, game
2. L-conv ($\mathbb{Z}^n$)	network tension, image processing OR (inventory, scheduling)
2. M-conv ($\mathbb{Z}^n$)	network flow, matching economics (game, auction) mixed polynomial matrix
3. M-conv (jump)	deg sequence, (2-)matching polynomial (half-plane property)
3. L-conv (graph)	multiflow, multifacility location

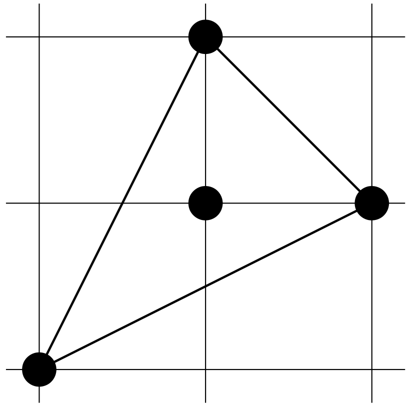
Integrally Convex Set $\subseteq \mathbb{Z}^n$



Integrally Convex Set

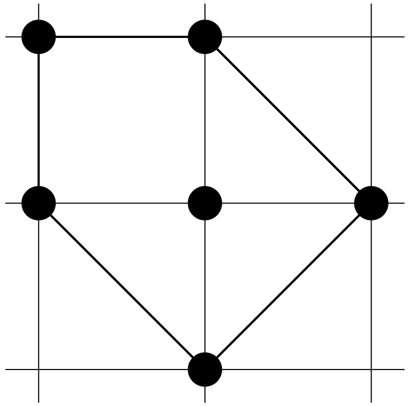


YES

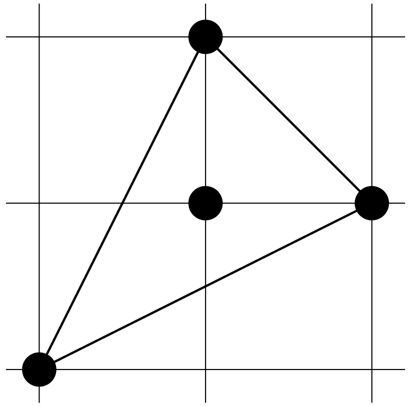


NO

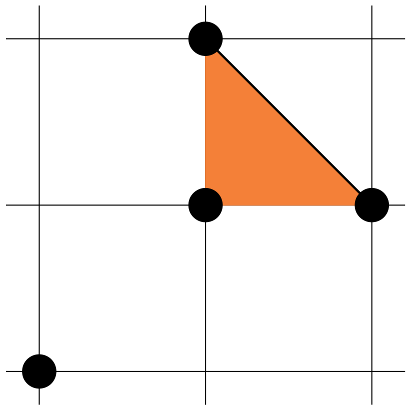
Integrally Convex Set



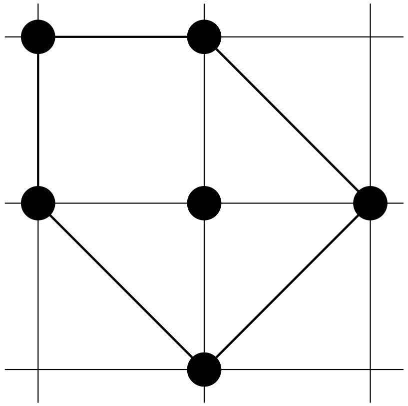
YES



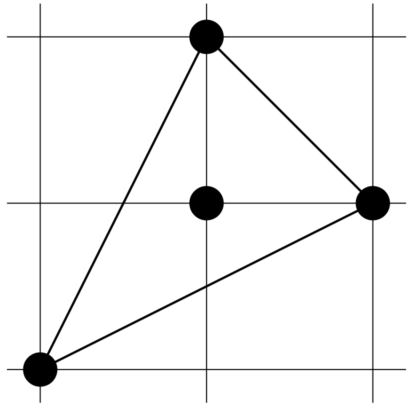
NO



Integrally Convex Set



YES



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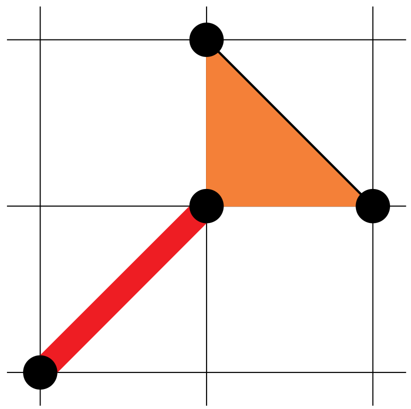
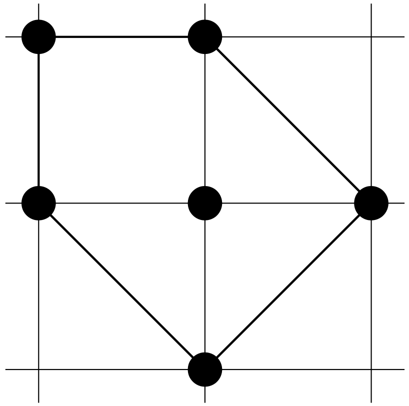
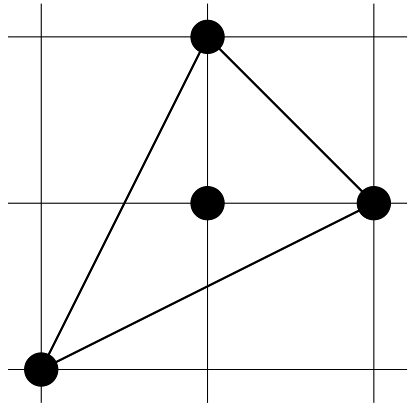
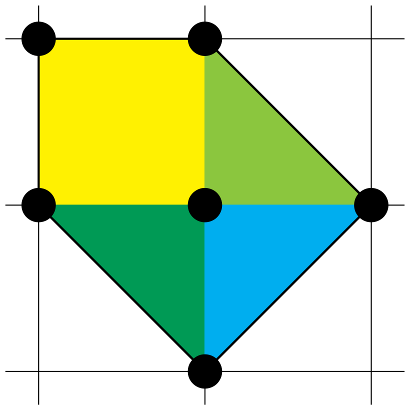


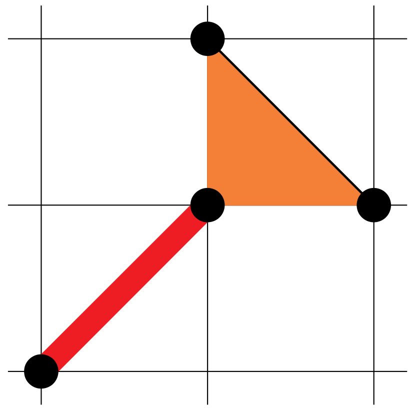
Integrally Convex Set





YES NO

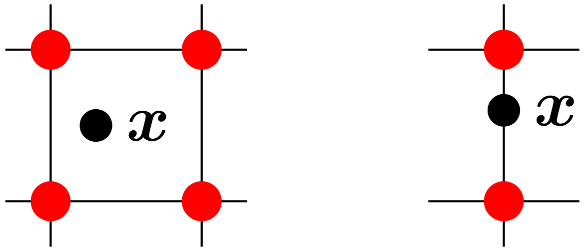




Integrally Convex Function

(Favati-Tardella 1990)

$N(x) = \{y \in \mathbb{Z}^n \mid \|x - y\|_\infty < 1\} \qquad (x \in \mathbb{R}^n)$

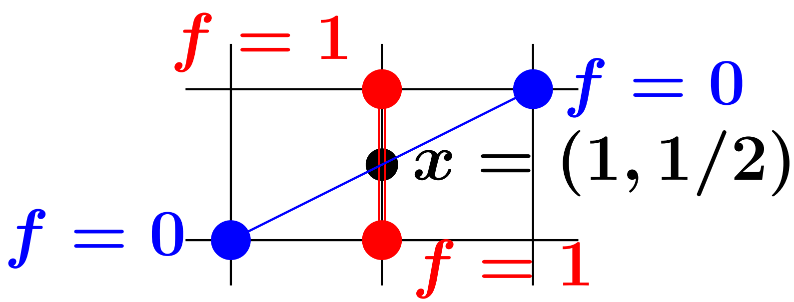


Local convex extension:

$$\tilde{f}(x) = \sup_{p,\alpha}\{\langle p, x\rangle + \alpha \mid \langle p, y\rangle + \alpha \le f(y) \ (\forall y \in N(x))\}$$

Def: f is integrally convex $\iff$ $\tilde{f}$ is convex

Ex: $f(x_1, x_2) = |x_1 - 2x_2|$ is NOT integrally convex



$\tilde{f}(x) = 1$

$\overline{f}(x) = 0$

Five Properties of “Convex” Functions

1. convex extension
2. local opt = global opt

hold, but

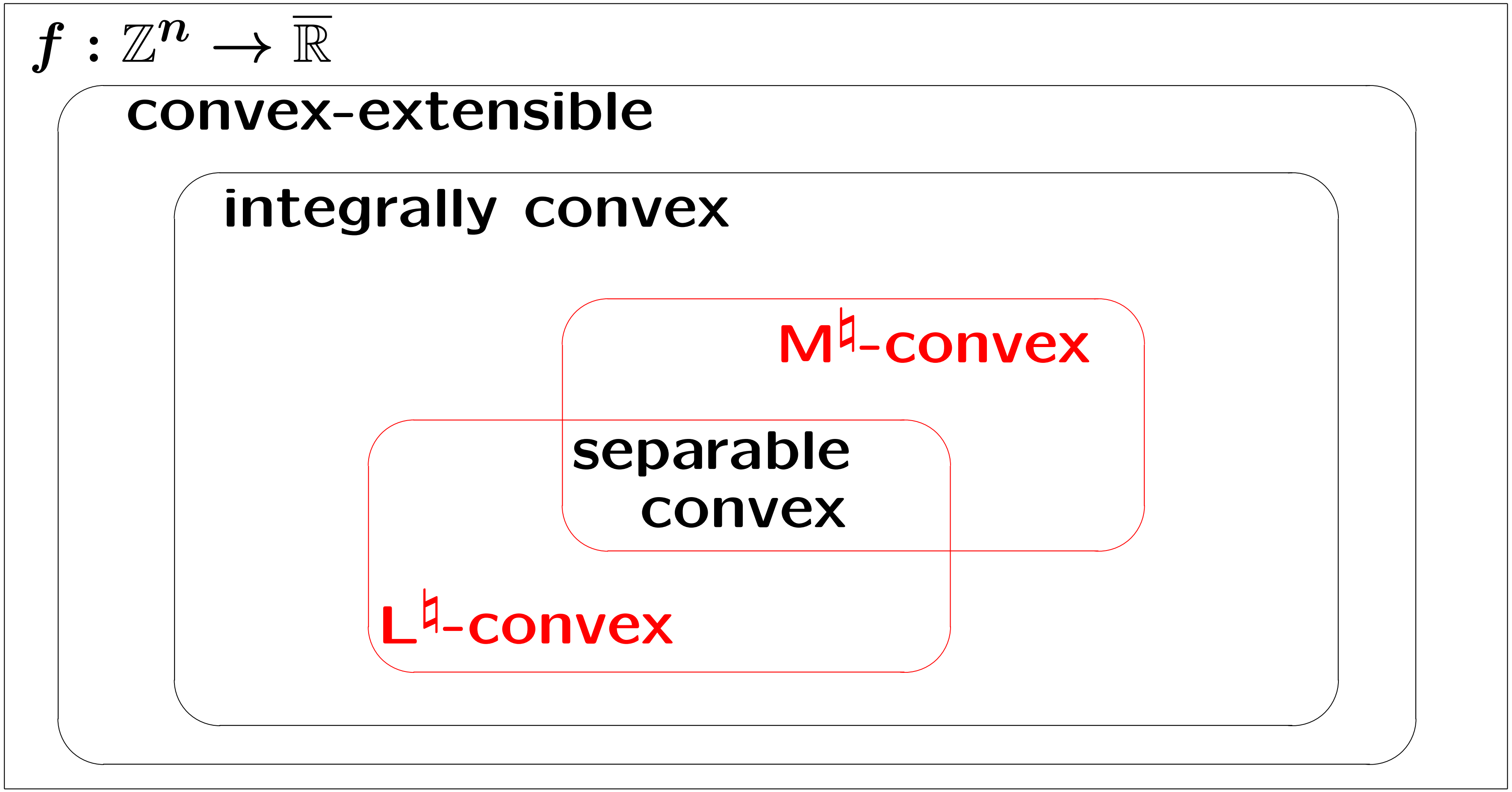
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

fail for **integrally convex functions**

Discrete Convex Functions

1. submodular (set fn)	✓
1. separable -conv	✓
1. integrally -conv	✓
2. L-conv($\mathbb{Z}^n$)	
2. M-conv($\mathbb{Z}^n$)	
3. M-conv(jump)	
3. L-conv(graph)	

Classes of Discrete Convex Functions



$\mathrm{M}^\natural \cap \mathrm{L}^\natural =$ separable

C3.

L-convex Functions

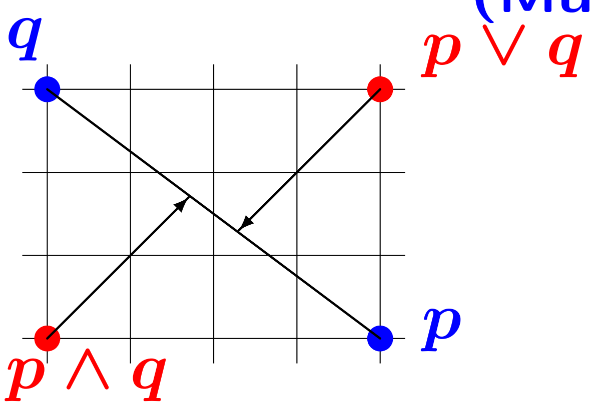
L-convex Function (L = Lattice)

(Murota 98)

$g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

$p \vee q$ compnt-max

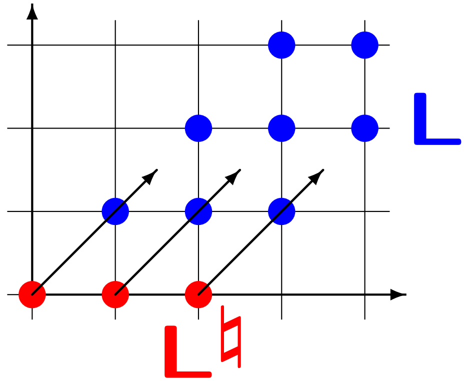
$p \wedge q$ compnt-min



Def: g is L-convex $\iff$

- Submodular: $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$
- Translation: $\exists r,\ \forall p$: $g(p + \mathbf{1}) = g(p) + r$

$$\mathbf{1} = (1, 1, \ldots, 1)$$



L♮-convexity from Submodularity

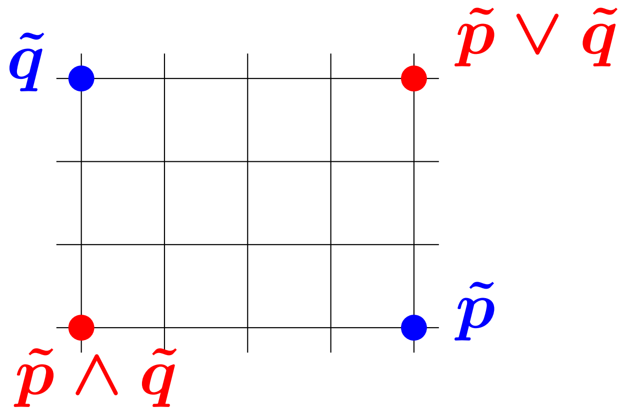
(Murota 98, Fujishige–Murota 2000)

$g : \mathbb{Z}^n \to \mathbb{R}$ **L♮-convex** $\iff$

$\tilde{g}(p_0, p) = g(p - p_0 \mathbf{1})$ is submodular in (p_0, p)

$\tilde{g} : \mathbb{Z}^{n+1} \to \mathbb{R}$, $\quad \mathbf{1} = (1, 1, \ldots, 1)$

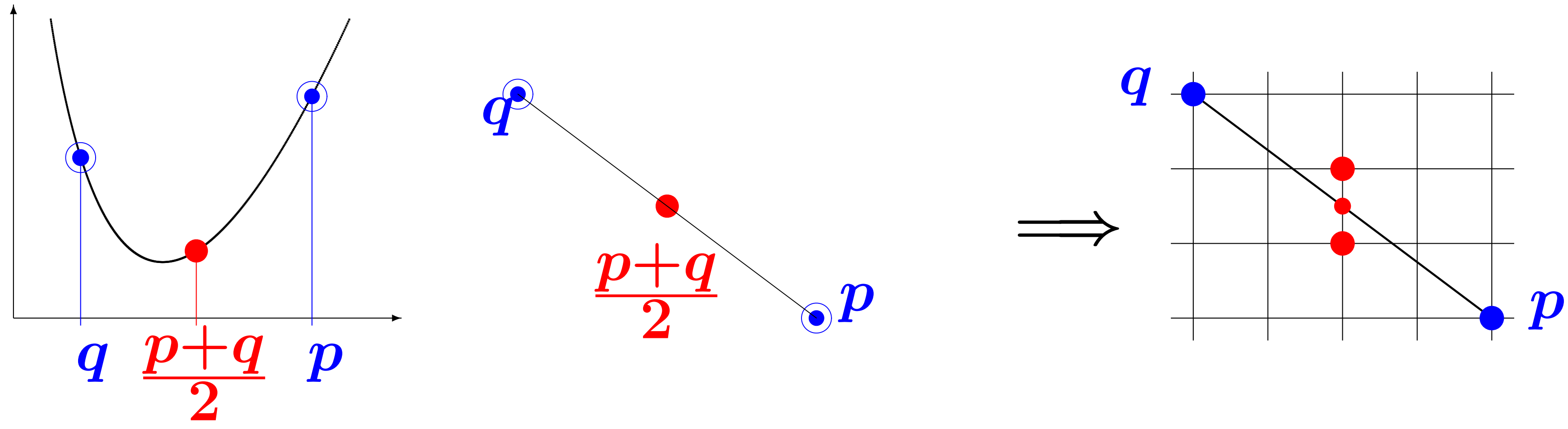
$\tilde{g}(\tilde{p}) + \tilde{g}(\tilde{q}) \geq \tilde{g}(\tilde{p} \vee \tilde{q}) + \tilde{g}(\tilde{p} \wedge \tilde{q})$



$\mathrm{L}_{n+1} \simeq \mathrm{L}^\natural_n \supsetneqq \mathrm{L}_n$

L♮-convexity from Mid-pt-convexity

(Favati-Tardella 1990, Fujishige–Murota 2000)



Mid-point convex ($g : \mathbb{R}^n \to \mathbb{R}$):

$$g(p) + g(q) \geq 2g\left(\frac{p+q}{2}\right)$$

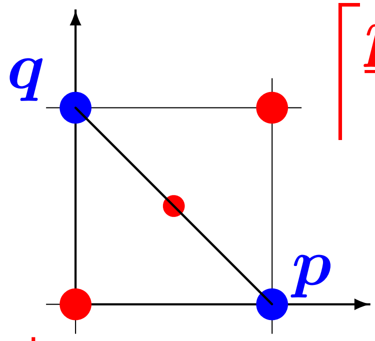
$\Longrightarrow$ **Discrete mid-point convex** ($g : \mathbb{Z}^n \to \mathbb{R}$)

$$g(p) + g(q) \geq g\left(\left\lceil\frac{p+q}{2}\right\rceil\right) + g\left(\left\lfloor\frac{p+q}{2}\right\rfloor\right)$$

L♮-convex function (L = Lattice)

Mid-pt Convexity for 01-Vectors

For $p, q \in \{0,1\}^n$



$$\left\lceil \frac{p+q}{2} \right\rceil = p \vee q$$

$$\left\lfloor \frac{p+q}{2} \right\rfloor = p \wedge q$$

Discrete mid-pt convexity:

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

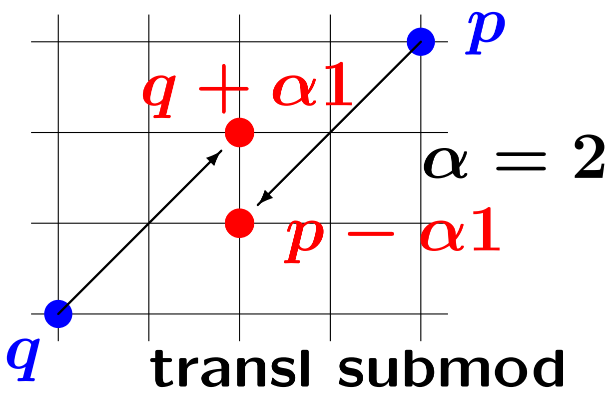
$\Longleftrightarrow$ Submodularity:

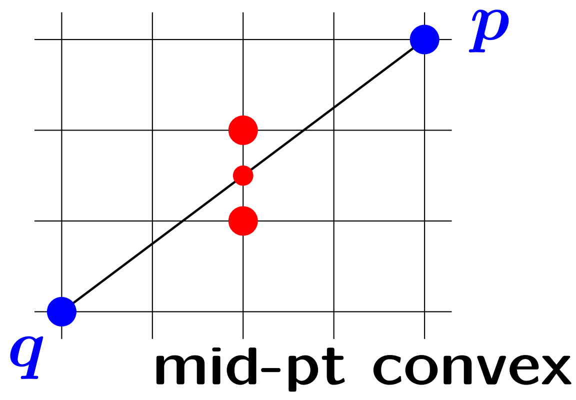
$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q)$$

Translation Submodularity (L$^\natural$)

$$g(p) + g(q) \geq g((p - \alpha 1) \vee q) + g(p \wedge (q + \alpha 1))$$

$(\alpha \geq 0)$





$\tilde{g}(p_0, p) = g(p - p_0 1)$ is submodular in (p_0, p)

(Fujishige-Murota 00)

$\Leftrightarrow$ translation submodular

(Fujishige-Murota 00)

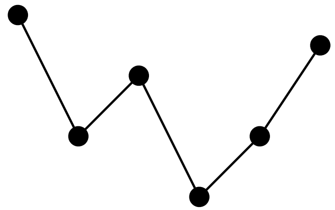
$\Leftrightarrow$ discrete mid-pt convex

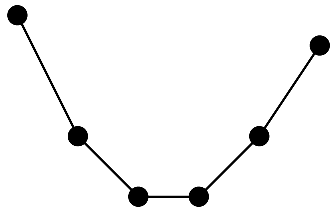
(Favati-Tardella 90)

$\Leftrightarrow$ submod. integ. convex

Rem: L$^\natural$-convex vs Submodular

$n = 1$





Fact 1: Every $g : \mathbb{Z} \to \mathbb{R}$ is submodular

Fact 2: $g : \mathbb{Z} \to \mathbb{R}$ is L$^\natural$-convex

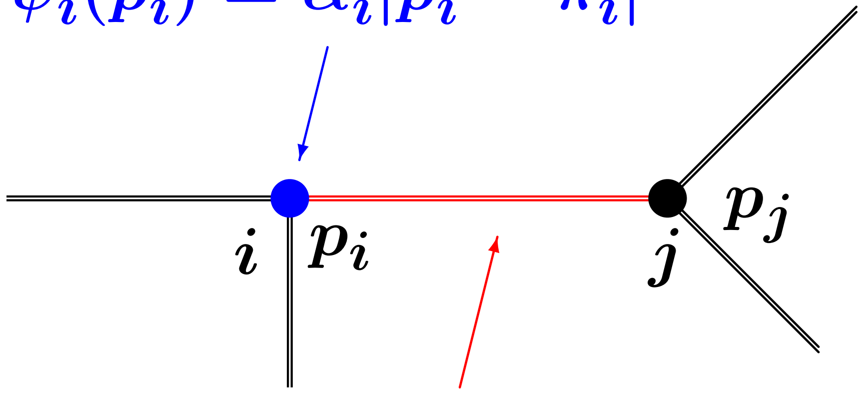
$\iff g(p-1) + g(p+1) \geq 2g(p)$ for all $p \in \mathbb{Z}$

Typical L$^\natural$-convex Function: Energy Function

π_i: given

$\alpha_i, \alpha_{ij} \geq 0$

$$\psi_i(p_i) = \alpha_i |p_i - \pi_i|^2$$



$$\psi_{ij}(p_i - p_j) = \alpha_{ij} |p_i - p_j|^2$$

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j) \quad \text{is L}^\natural\text{-convex}$$

ψ_i, ψ_{ij}: any univariate convex functions

L$^\natural$-convex Function: Examples

Quadratic: $g(p) = \sum_i \sum_j a_{ij} p_i p_j$ is L$^\natural$-convex

$$\Leftrightarrow \ a_{ij} \leq 0 \quad (i \neq j), \qquad \sum_j a_{ij} \geq 0 \ (\forall i)$$

Energy function: For univariate convex ψ_i and ψ_{ij}

$$g(p) = \sum_i \psi_i(p_i) + \sum_{i \neq j} \psi_{ij}(p_i - p_j)$$

Range: $g(p) = \max\{p_1, p_2, \ldots, p_n\} - \min\{p_1, p_2, \ldots, p_n\}$

Submodular set function: $\rho : 2^V \to \overline{\mathbb{R}}$

$$\Leftrightarrow \ \rho(X) = g(\chi_X) \quad \text{for some L}^\natural\text{-convex } g$$

Multimodular: $h : \mathbb{Z}^n \to \overline{\mathbb{R}}$ is multimodular $\Leftrightarrow$

$h(p) = g(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ for L$^\natural$-convex g

Five Properties of "Convex" Functions

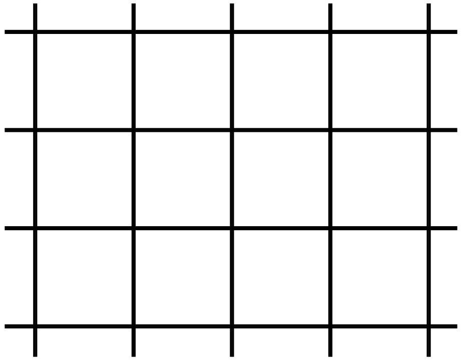
1. convex extension
2. local opt $=$ global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **L-convex functions**

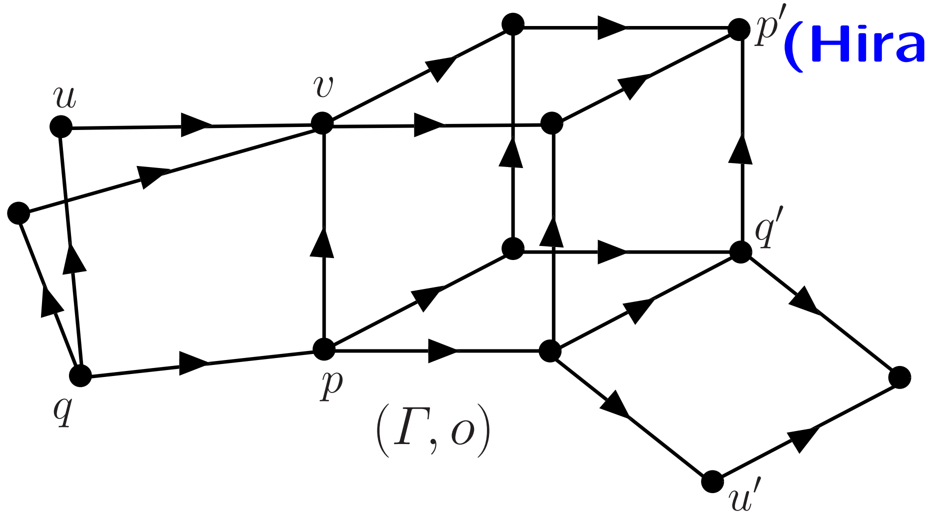
$\Rightarrow$ **Part II**

L-convex Function on Graphs

(Hirai 12)

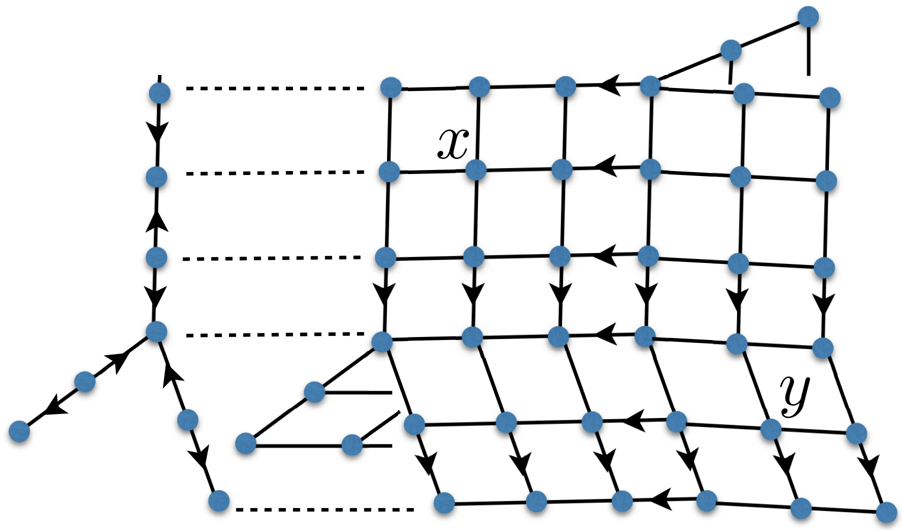


integer lattice



oriented modular graph

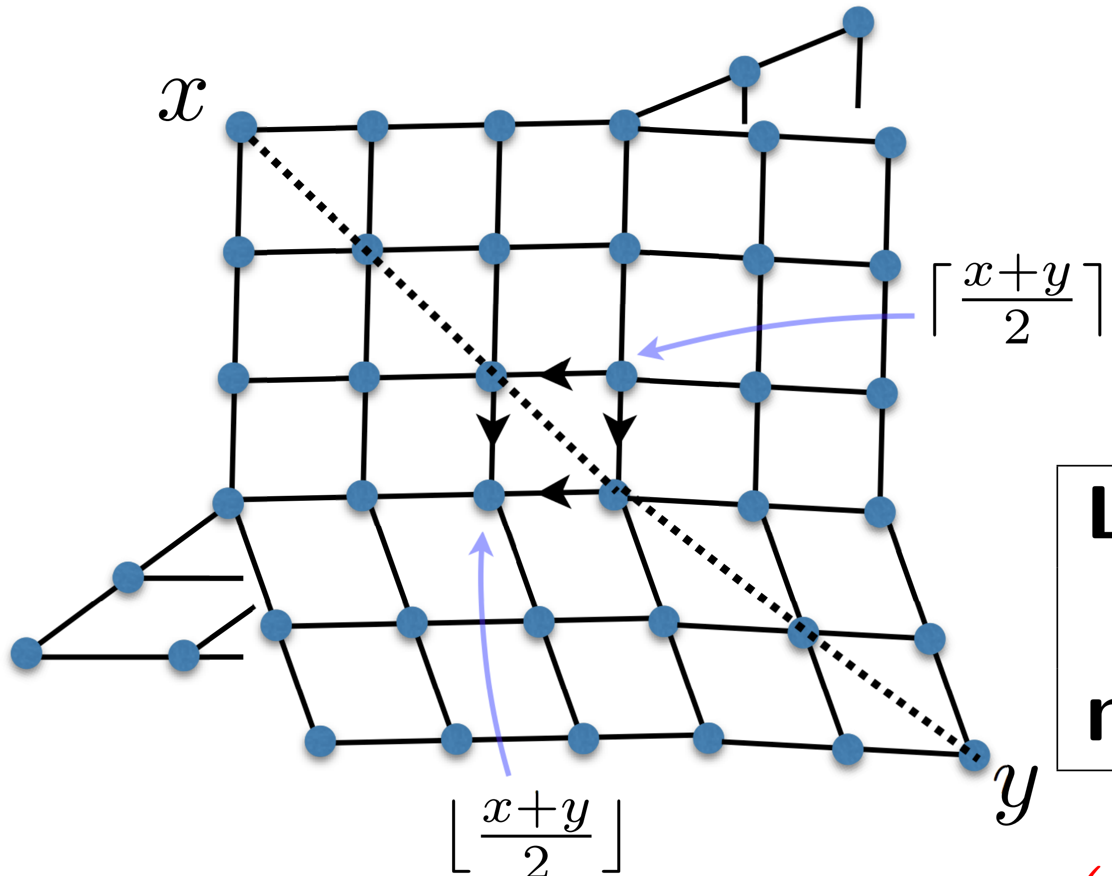
direct product of trees:



(Kolmogorov 11)
(Huber-Kolmogorov 12)

Mid-point Convexity on Tree Products

(Hirai 13,15)



L-convex
|| (def)
mid-point convex

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right)$$

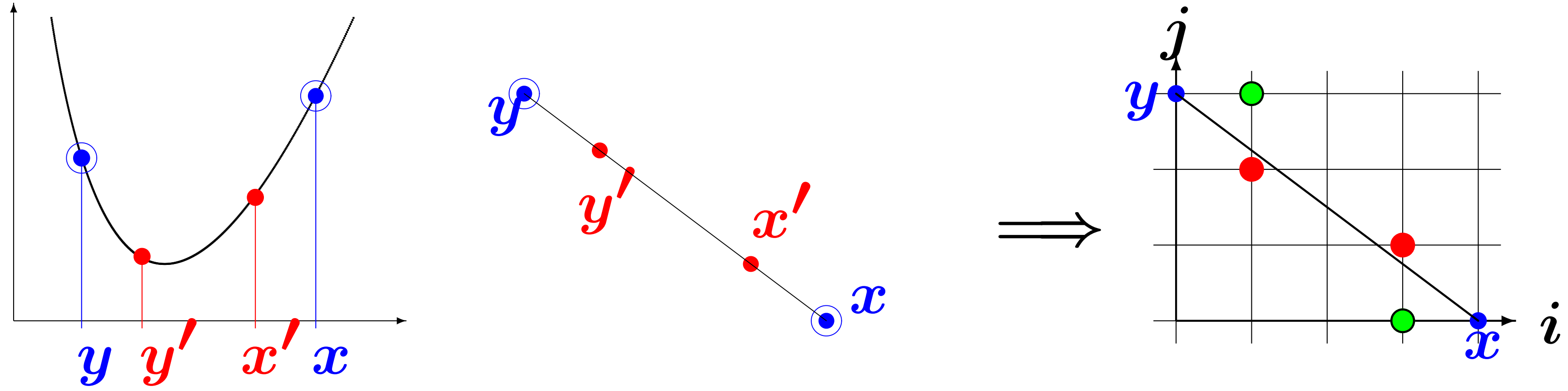
- submodular on (rooted) trees (Kolmogorov 11)
- k-submodular (Huber-Kolmogorov 12)

C4.

M-convex Functions

M♮-convexity from Equi-dist-convexity

(Murota 1996, Murota−Shioura 1999)



Equi-distance convex ($f : \mathbb{R}^n \to \mathbb{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange** ($f : \mathbb{Z}^n \to \mathbb{R}$) $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \Big[f(x - e_i) + f(y + e_i),$$

$$\min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \Big]$$

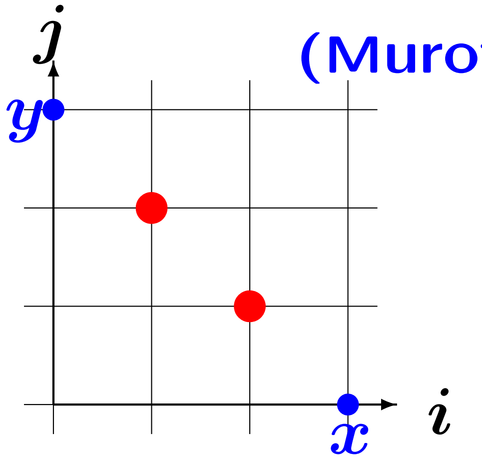
M♮-convex function $\qquad$ (M = Matroid)

M-convex Function (M = Matroid)

$f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$

e_i: i-th unit vector

(Murota 96)

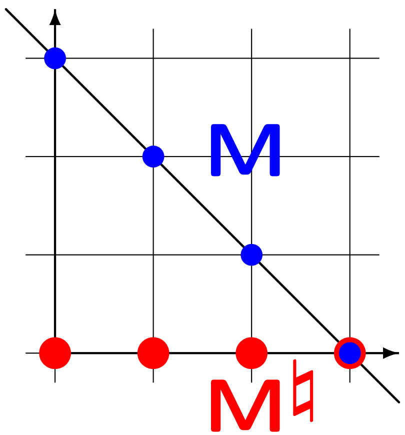


Def: f is M-convex

$\iff \quad \forall x, y, \quad \forall i : x_i > y_i, \quad \exists j : x_j < y_j$:

$$f(x) + f(y) \geq f(x - e_i + e_j) + f(y + e_i - e_j)$$

$\mathrm{dom}\, f \subseteq$ const-sum hyperplane



$$\mathrm{M}_{n+1} \simeq \mathrm{M}^{\natural}_n \supsetneq \mathrm{M}_n$$

M$^\natural$-convex Function: Examples

Quadratic: $f(x) = \sum a_{ij} x_i x_j$ is M$^\natural$-convex

$$\Leftrightarrow a_{ij} \geq 0, \quad a_{ij} \geq \min(a_{ik}, a_{jk}) \ (\forall k \not\in \{i,j\})$$

Min value: $f(X) = \min\{a_i \mid i \in X\}$ [unit preference]

Cardinality convex: $f(X) = \varphi(|X|)$ (φ: convex)

Separable convex: $f(x) = \sum \varphi_i(x_i)$ (φ_i: convex)

Laminar convex: $f(x) = \sum_A \varphi_A(x(A))$ (φ_A: convex)

$\{A, B, ...\}$: laminar $\Leftrightarrow$ $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

M$^\natural$-concave Functions from Matroids

Matroid rank: $f(X) = r(X)$ (rank of X) (Fujishige 05)

Matroid rank sum: $f(X) = \sum \alpha_i r_i(X)$

$r_i \leftarrow r_{i+1}$ (strong quotient), $\alpha_i \geq 0$ (Shioura 12)

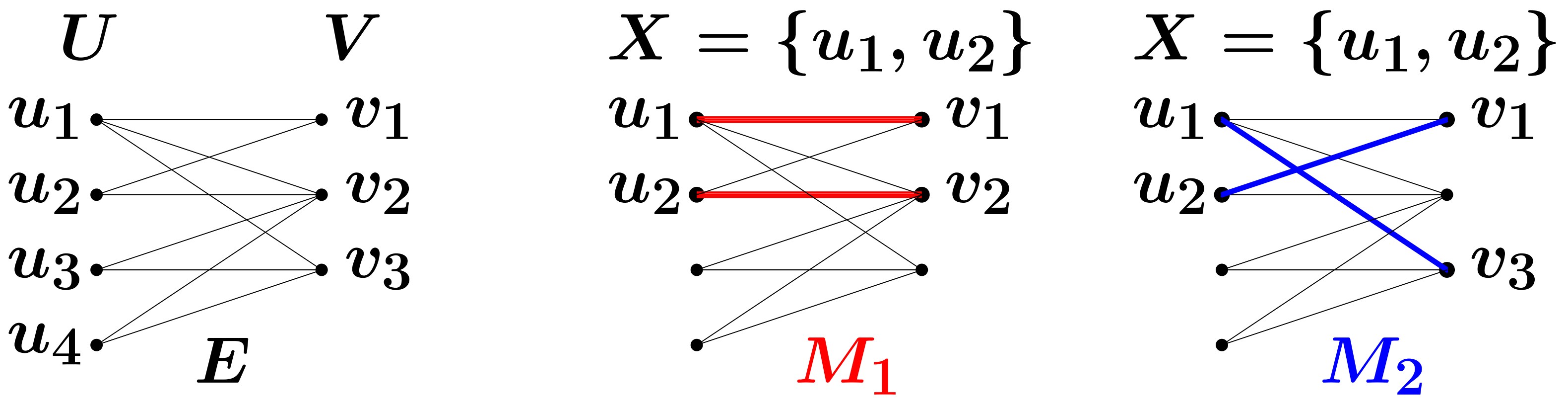
Weighted matroid: w: weight vector

$f(X) = \max\{w(Y) \mid Y\text{: indep} \subseteq X\}$ (Shioura 12)

Valuated matroid: $\omega : 2^V \to \underline{\mathbb{R}}$

$\Leftrightarrow \omega(X) = f(\chi_X)$ for some M-concave f

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{\sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X\}$$

Max-weight func f is M$^\natural$-concave (Murota 1996)

- Proof by augmenting path
- Extension to min-cost network flow

Polynomial Matrix

(Dress-Wenzel 90)
Valuated Matroid

$$A = \begin{array}{|c|c|c|c|} \hline s+1 & s & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \hline \end{array} \qquad \omega(J) = \deg \det A[J]$$

$\mathcal{B} = \{J \mid J$ is a base of column vectors$\}$

Grassmann-Plücker $\Rightarrow$ Exchange (M-concave)

For any $J, J' \in \mathcal{B}$, $i \in J \setminus J'$, there exists $j \in J' \setminus J$
s.t. $J - i + j \in \mathcal{B}$, $J' + i - j \in \mathcal{B}$,

$$\omega(J) + \omega(J') \leq \omega(J - i + j) + \omega(J' + i - j)$$

Ex. $J = \{1, 2\}$, $\quad J' = \{3, 4\}$, $\quad i = 1$

$\det A[\{1,2\}] = \det A[\{3,4\}] = 1$, $\quad \omega(J) = \omega(J') = 0$

Can take $j = 3$: $\quad J - i + j = \{3, 2\}$, $\; J' + i - j = \{1, 4\}$

$\omega(J - i + j) = 1$, $\omega(J' + i - j) = 1$

Five Properties of "Convex" Functions

1. convex extension
2. local opt = global opt
3. conjugacy (Legendre transform)
4. separation theorem
5. Fenchel duality

hold for **M-convex functions**

$\Rightarrow$ **Part II**

Five Properties (Summary/Preview)

	convex ext.	local opt/ global opt	Legendre biconjug.	separat. theorem	Fenchel duality
submod. (set fn)	Y	Y	Y	Y	Y
separable -convex	Y	Y	Y	Y	Y
integrally -convex	Y	Y	N	N	N
L-convex ($\mathbb{Z}^n$)	Y	Y	Y	Y	Y
M-convex ($\mathbb{Z}^n$)	Y	Y	Y	Y	Y

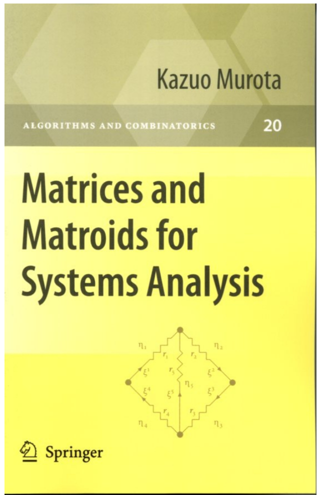
Books (discrete convex analysis)

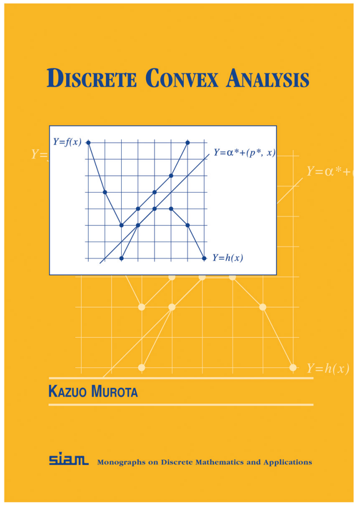
2000: Murota, Matrices and Matroids for Systems Analysis, Springer

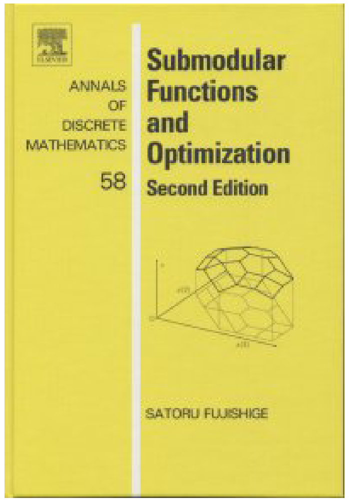
2003: Murota, Discrete Convex Analysis, SIAM

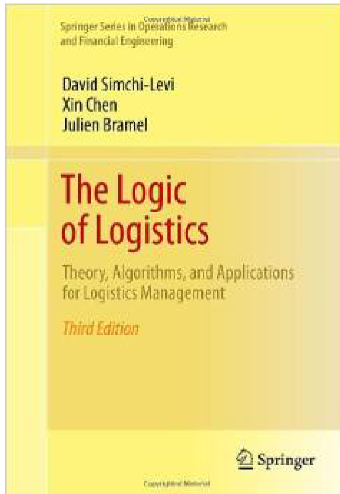
2005: Fujishige, Submodular Functions and Optimization, 2nd ed., Elsevier

2014: Simchi-Levi, Chen, Bramel, The Logic of Logistics, 3rd ed., Springer









Books (in Japanese)

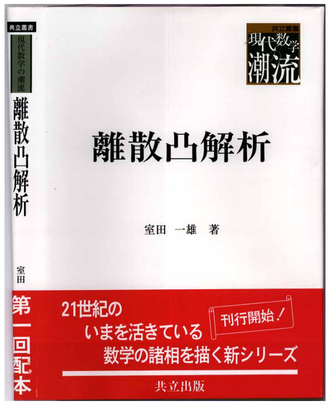
2001: Murota，離散凸解析，共立出版
(Discrete Convex Analysis)

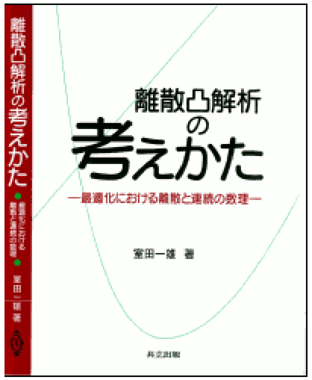
2007: Murota，離散凸解析の考えかた，共立出版
(Primer of Discrete Convex Analysis)

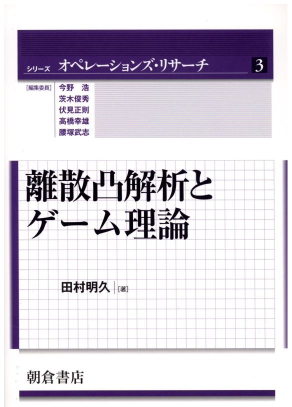
2009: Tamura，離散凸解析とゲーム理論，朝倉書店
(Discrete Convex Analysis and Game Theory)

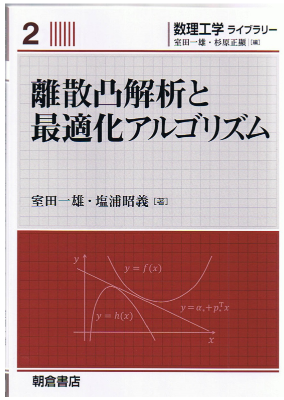
2013: Murota,Shioura,離散凸解析と最適化アルゴリズム，朝倉
(Discrete Convex Analysis and Optimization Algorithms)

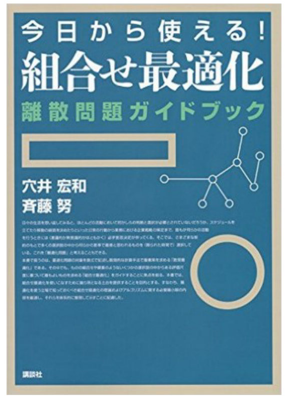
2015: Anai，Saito，今日から使える！組合せ最適化，講談社
(A Guidebook of Combinatorial Optimization)











Survey/Slide/Video/Software

[Survey]
Murota: Recent developments in discrete convex analysis (Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, 219–260)

[Slide]

http://www.comp.tmu.ac.jp/kzmurota/publist.html#DCA

[Video]

https://smartech.gatech.edu/xmlui/handle/1853/43257/

https://smartech.gatech.edu/xmlui/handle/1853/43258/

[Software] DCP (Discrete Convex Paradigm)
http://www.misojiro.t.u-tokyo.ac.jp/DCP/

END