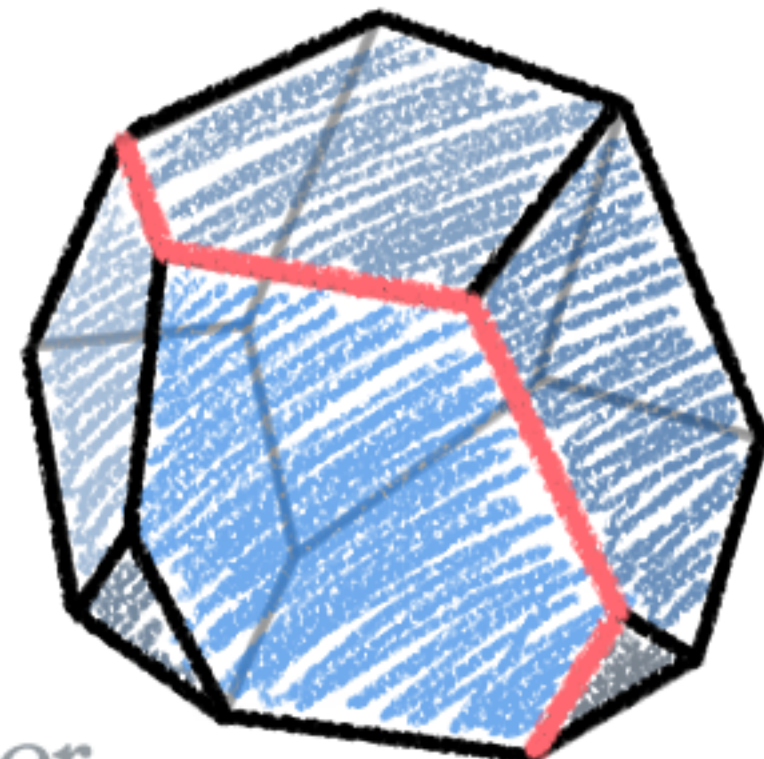
Algorithmic and geometric aspects of combinatorial and continuous optimization



Antoine Deza, McMaster

based on joint works with

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George Manoussakis, Orsay

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Advanced Optimization Laboratory

linear optimization

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ find, in any, a d-dimensional vector $\boldsymbol{x}$ such that :

$\boldsymbol{Ax} = \boldsymbol{b}$	$\boldsymbol{Ax} = \boldsymbol{b}$ $\boldsymbol{x} \geq \mathbf{0}$
linear algebra	**linear optimization**

*"Can **linear optimization** be solved in **strongly polynomial*** time?" is listed by Smale (Fields Medal 1966) as one of the top problems for the XXI century

Polynomial : execution time bounded by a ***polynomial*** in n, d, and ***input data length L***

linear optimization

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** find, in any, a ***d***-dimensional vector ***x*** such that :

$\boldsymbol{Ax} = \boldsymbol{b}$	$\boldsymbol{Ax} = \boldsymbol{b}$
	$\boldsymbol{x} \geq \mathbf{0}$
linear algebra	**linear optimization**

*"Can **linear optimization** be solved in **strongly polynomial*** time?" is listed by Smale (Fields Medal 1966) as one of the top problems for the XXI century

Strongly polynomial : ***polynomial*** time; number of arithmetic operations bounded by a polynomial in the ***dimension*** of the problem (***independent*** from the ***input data length L***)

linear optimization algorithms

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** and a ***d***-dimensional cost vector ***c***, solve : { max $\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ }

Simplex methods (Dantzig 1947) pivot-based, combinatorial, ***not proven to be polynomial***, efficient in practice

Ellipsoid methods (Khachiyan 1979)
polynomial ⇒ ***linear optimization is polynomial time solvable***

Interior point methods (Karmarkar 1984)
path-following, ***polynomial***, efficient in practice

Primal-dual interior point (Kojima-Mizuno-Yoshise 1989)

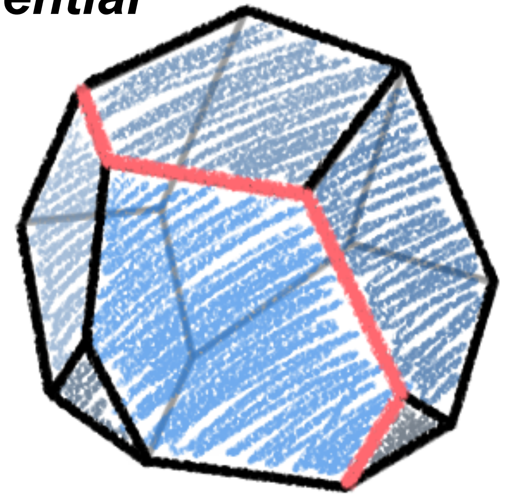
Criss-cross (Terlaky 1983, Wang 1985, Chang 1979)
Volumetric (Vaidya-Atkinson 1993, Anstreicher 1997)
Monotonic build-up simplex (Anstreicher-Terlaky 1994)
…..

linear optimization algorithms
simplex methods

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** and a ***d***-dimensional cost vector ***c***, solve : { max $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$: $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{x} \geq \mathbf{0}$ }

Simplex methods (Dantzig 1947): pivot-based, combinatorial, ***not proven to be polynomial***, efficient in practice

- start from a ***feasible basis***
- use a ***pivot rule***
- find an optimal solution after a ***finite number*** of iterations
- most known pivot rules are known to be ***exponential*** (worst case); ***efficient*** implementations exist



Given a … atrix ***A***
and a *c* … x ≥ **0** }

Simple … ***not***
proven

- start
- use
- find
- mos
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How Good Is the Simplex Algorithm?

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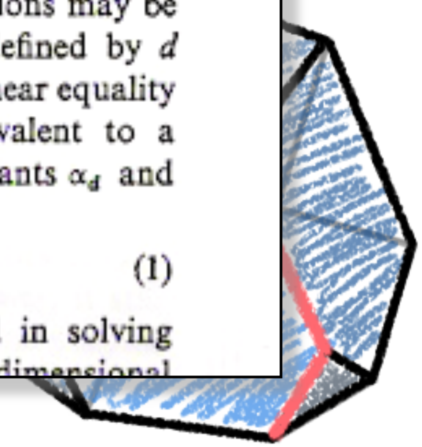
1. INTRODUCTION

By constructing long "increasing" paths on appropriate convex polytopes, we show that the simplex algorithm for linear programs (at least with its most commonly used pivot rule, Dantzig [1]) is not a "good algorithm" in the sense of Jack Edmonds. That is, the number of pivots or iterations that may be required is not majorized by any polynomial function of the two parameters that specify the size of the program. In particular, $2^d - 1$ iterations may be required in solving a linear program whose feasible region, defined by d linear inequality constraints in d nonnegative variables or by d linear equality constraints in $2d$ nonnegative variables, is projectively equivalent to a d-dimensional cube. Further, for each d there are positive constants α_d and β_d such that

$$\alpha_d n^{[d/2]} < \Xi(d, n) < \beta_d n^{[d/2]} \qquad \text{for all} \quad n > d, \qquad (1)$$

where $\Xi(d, n)$ is the maximum number of iterations required in solving nondegenerate linear programs whose feasible regions are d-dimensional



linear optimization algorithms
simplex methods

Klee-Minty 1972: edge-path followed by the simplex method with Dantzig's rule visits the 2^{d} vertices of a ***combinatorial*** cube ($n = 2d$)
$\Rightarrow 2^{d} - 1$ pivots required to reach the optimum

Zadeh 1973 : bad network problems

Zadeh 1980 : *deformed products* and *least entered* rule

Amenta-Ziegler 1999 : *deformed products*

Friedmann 2011 : *least entered* rule *is superpolynomial*

Surveys : Terlaky-Zhang 1993, Ziegler 2004, Meunier 2013

… Avis-Friedmann 2016…

Dear Victor,

Please post this offer of $1000 to the first person who can find a counterexample to the least entered rule or prove it to be polynomial. The least entered rule enters the improving variable which has been entered least often.

Sincerely,

Norman Zadeh

Zadeh's offer (Ziegler 2004)
(Avis' postface to Zadeh 1980 report, 2009 reprint)

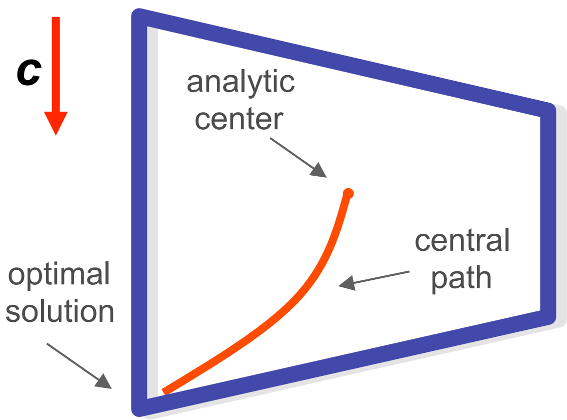
David Avis, Norman Zadeh, Oliver Friedmann, Russ Caflish (IPAM 2011)

linear optimization algorithms
(central path following) interior point methods

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** and a ***d***-dimensional cost vector ***c***, solve : $\{ \max \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \}$

Interior Point Methods :
path-following, ***polynomial***, efficient in practice

- start from the ***analytic center***
- follow the ***central path***
- converge to an optimal solution in $O(\sqrt{n}L)$ iterations
  (L: input data length)



$$\min \quad \boldsymbol{c}^{\mathrm{T}}x - \boldsymbol{\mu} \sum_i \ln(b - Ax)_i$$

μ : central path parameter
$x \in \boldsymbol{P} : Ax \leq b$

linear optimization
(some) combinatorial and geometric parameters

Tardos 1985: algorithm polynomial in $\boldsymbol{n}$, $\boldsymbol{d}$, and $\boldsymbol{L_A}$ (size of $\boldsymbol{A}$)
⇒ strongly polynomial for minimum cost flow, bipartite matching etc.
… Orlin 1986, Kitahara-Mizuno 2011, Mizuno 2014, Mizuno-Sukegawa-Deza 2015...

Ye 2011 : strongly polynomial simplex for Markov Decision Problem

Vavasis-Ye 1996 : $O(\boldsymbol{d}^{3.5} \log(\boldsymbol{d}\, \chi_{\boldsymbol{A}}))$ primal-dual interior point method
…Megiddo-Mizuno-Tsuchiya 1998, Monteiro-Tsuchiya 2003…

Bonifas-Summa-Eisenbrand-Hähnle-Niemeier 2014: $O(\boldsymbol{d}^4 \Delta_{\boldsymbol{A}}^2 \log(\boldsymbol{d}\, \Delta_{\boldsymbol{A}}))$ diameter ($\Delta_{\boldsymbol{A}}$ largest sub-determinant norm; Dyer-Frieze 1994)

Dadush-Hähnle 2015: $O(\boldsymbol{d}^3/\delta_{\boldsymbol{A}} \log(\boldsymbol{d}/\delta_{\boldsymbol{A}}))$ expected (shadow vertex) simplex pivots ($\delta_{\boldsymbol{A}}$ *curvature* ; $1/\delta_{\boldsymbol{A}} \leq \boldsymbol{d}\, \Delta_{\boldsymbol{A}}^2$)

….

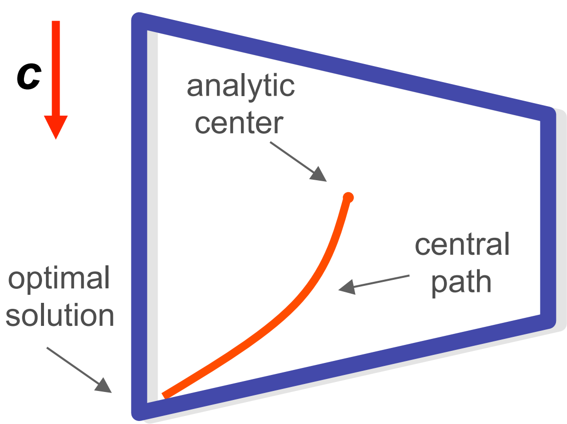
linear optimization diameter and curvature

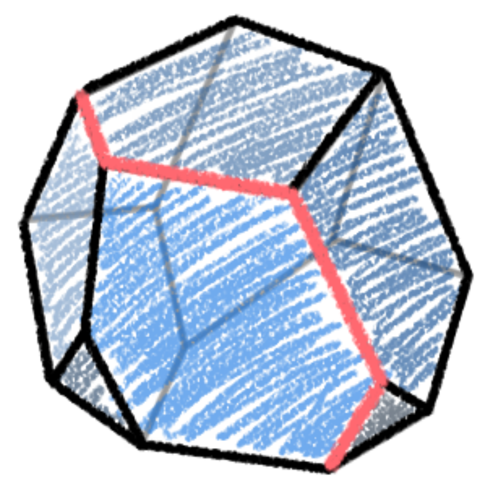
Diameter (*of a polytope*) :

lower bound for the number of iterations for *pivoting* **simplex methods**

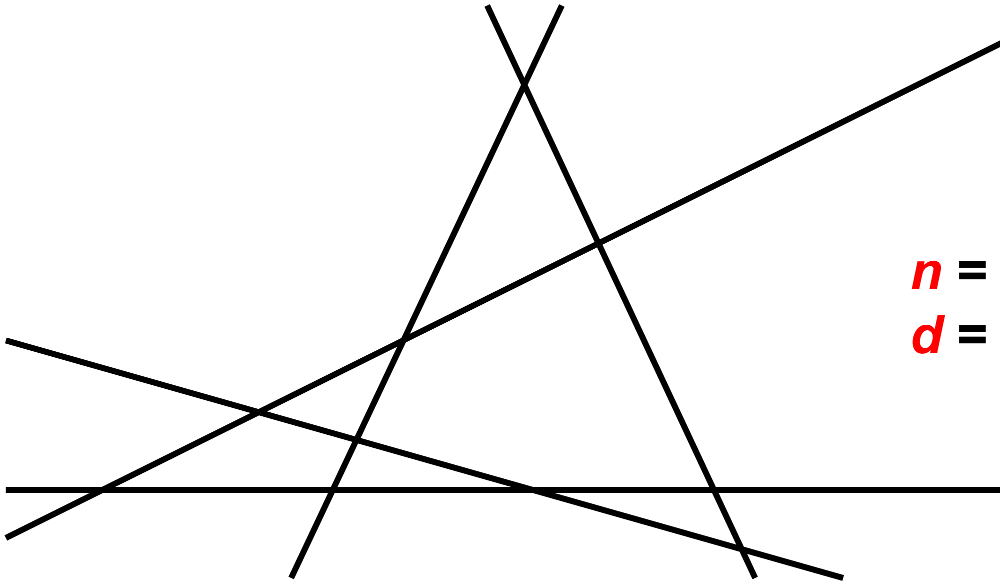
Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iterations for *path following* **interior point methods**





linear optimization : diameter and curvature

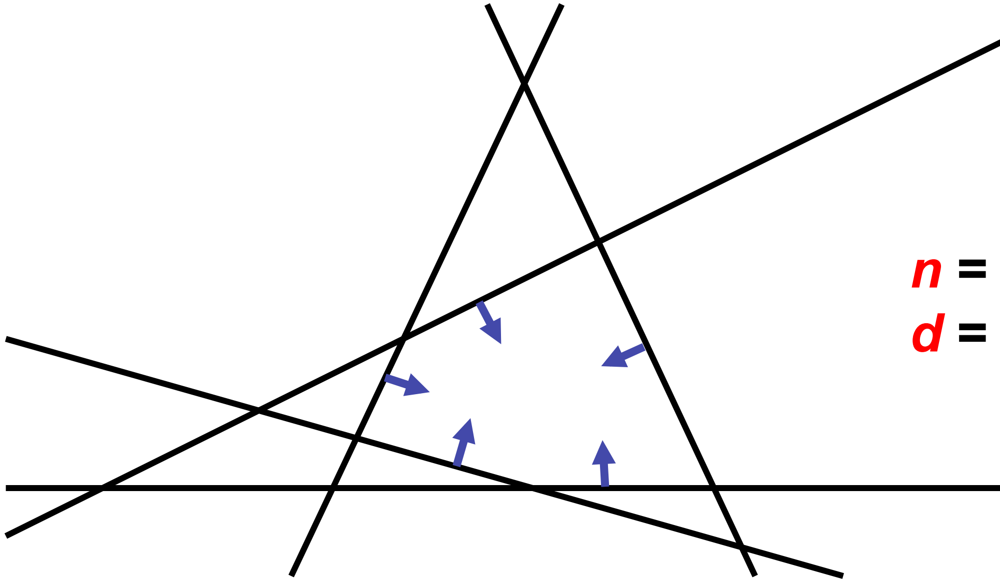


***n* = 5** : *inequalities*
***d* = 2** : *dimension*

Polytope *P* defined by ***n*** **inequalities** in **dimension *d***

❖ polytope : *bounded* polyhedron

linear optimization : diameter and curvature



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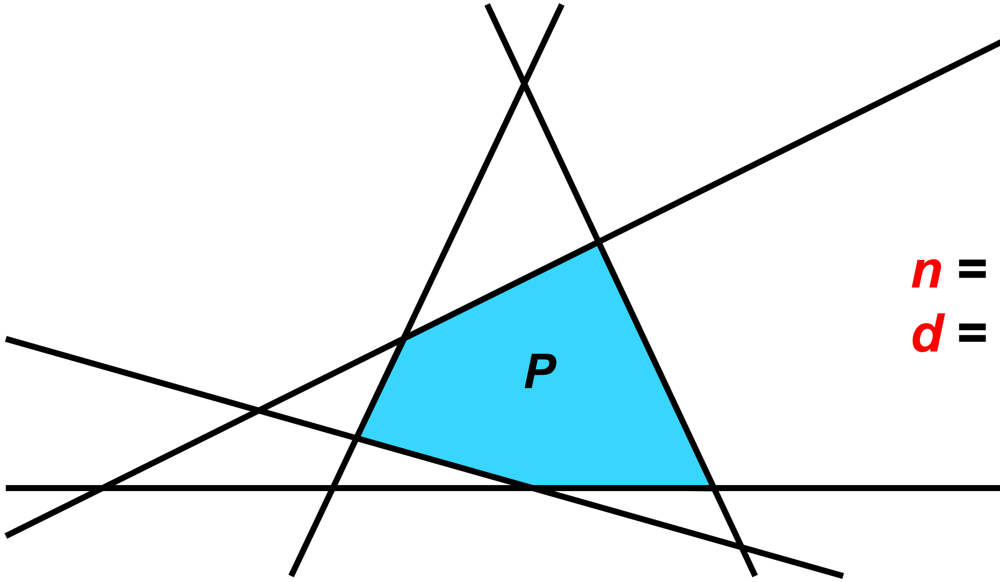
linear optimization : diameter and curvature



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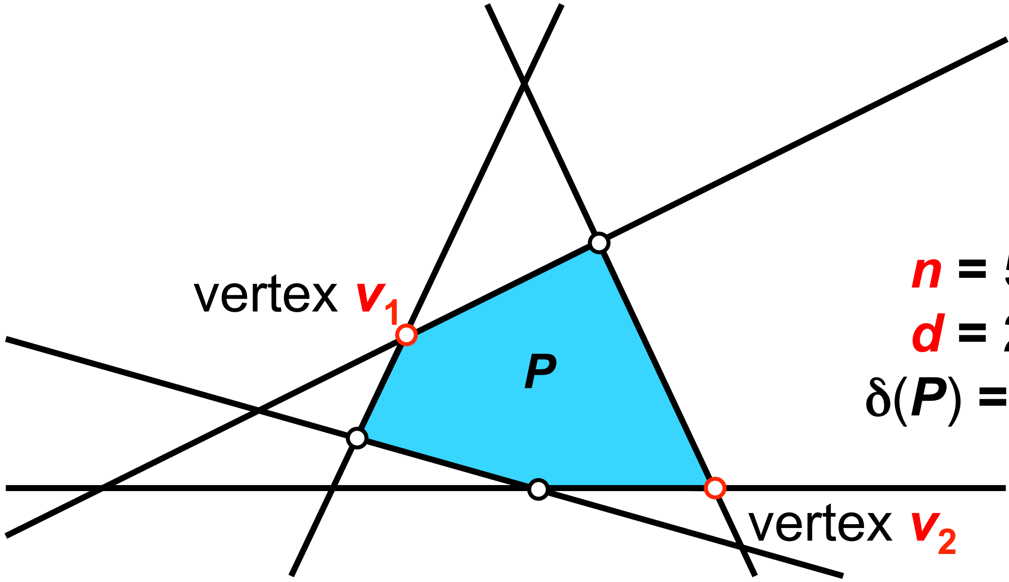
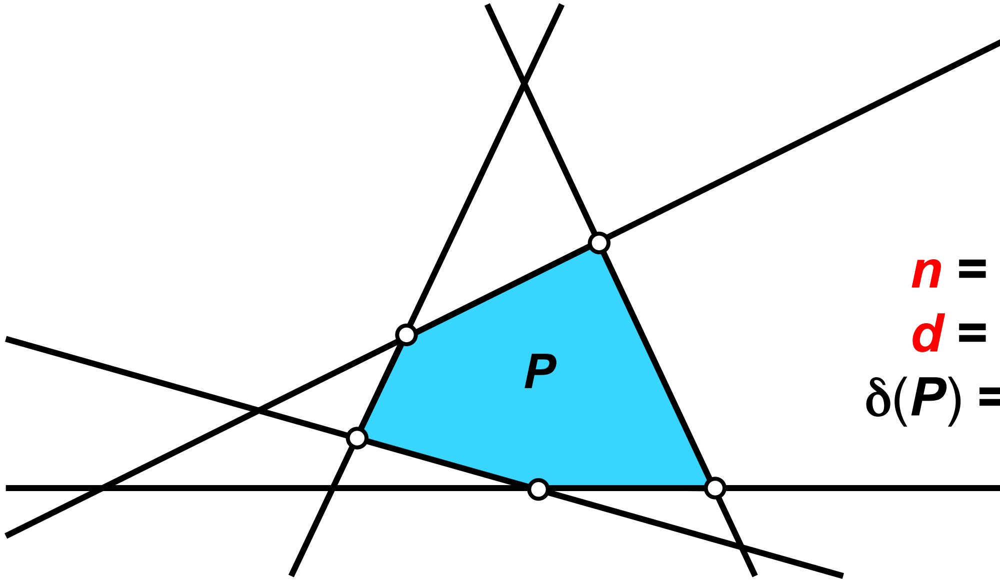
linear optimization : diameter and curvature



n **= 5** : *inequalities*
d **= 2** : *dimension*
$\delta(P)$ **= 2** : *diameter*

Diameter $\delta(P)$: smallest number such that **any two vertices** (v_1,v_2) can be connected by a **path with at most** $\delta(P)$ **edges**

linear optimization : diameter and curvature

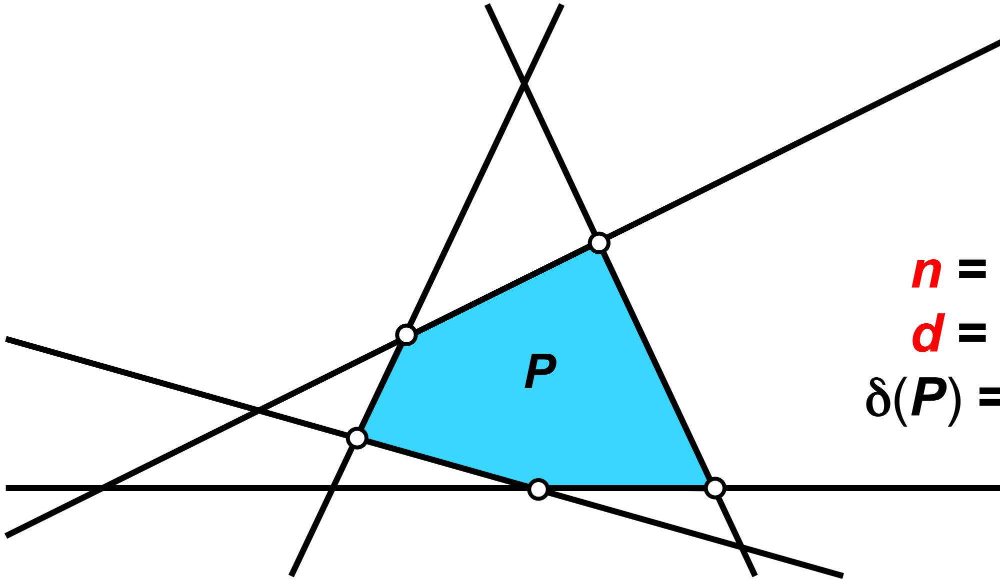


$\boldsymbol{n} = \mathbf{5}$: *inequalities*
$\boldsymbol{d} = \mathbf{2}$: *dimension*
$\delta(\boldsymbol{P}) = \mathbf{2}$: *diameter*

Diameter $\delta(P)$: smallest number such that any two vertices can be connected by a path with at most $\delta(P)$ edges

Hirsch Conjecture 1957 : $\delta(P) \leq n - d$

linear optimization : diameter and curvature



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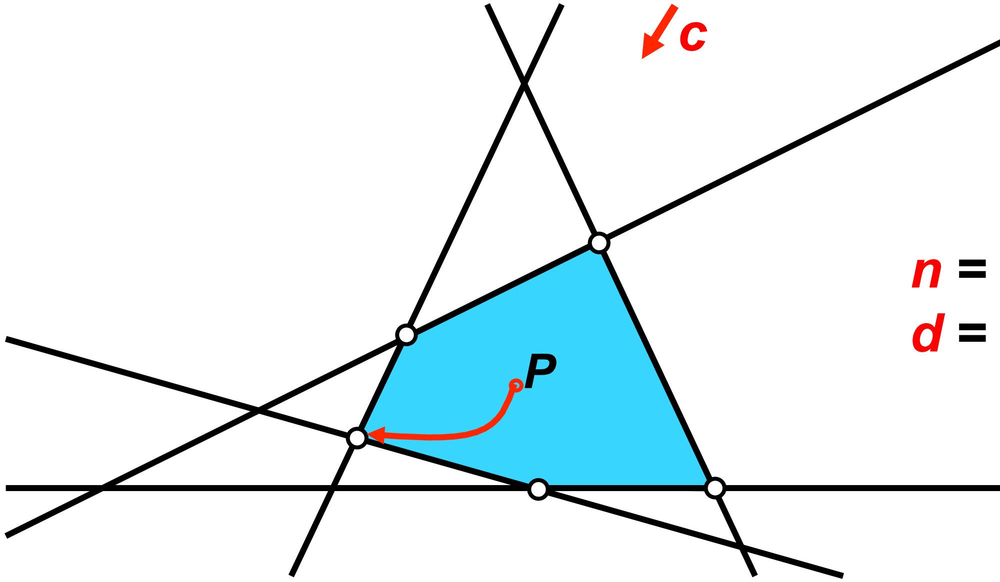
Diameter $\delta(P)$: smallest number such that any two vertices can be connected by a path with at most $\delta(P)$ edges

Hirsch Conjecture 1957 : $\delta(P) \leq n - d$

➢ ***disproved*** by Santos 2012 using construction with $n = 2d$

linear optimization : diameter and curvature

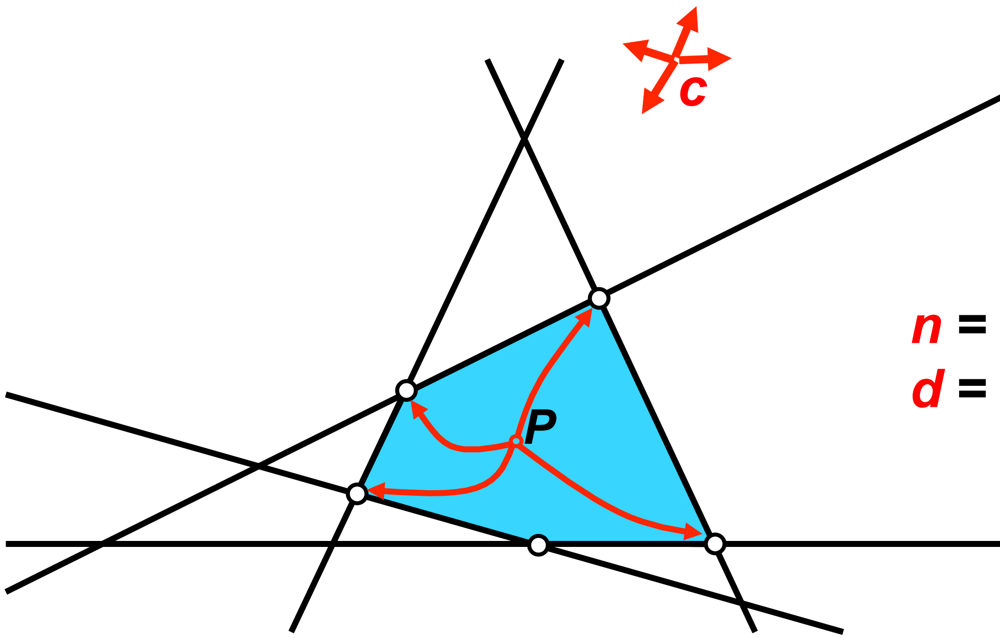


***n* = 5** : *inequalities*
***d* = 2** : *dimension*

$\lambda^{c}(P)$: total **curvature** of the primal central path of $\{ \max c^{T}x : x \in P \}$

❖ $\lambda^{c}(P)$: *redundant* inequalities count

linear optimization : diameter and curvature



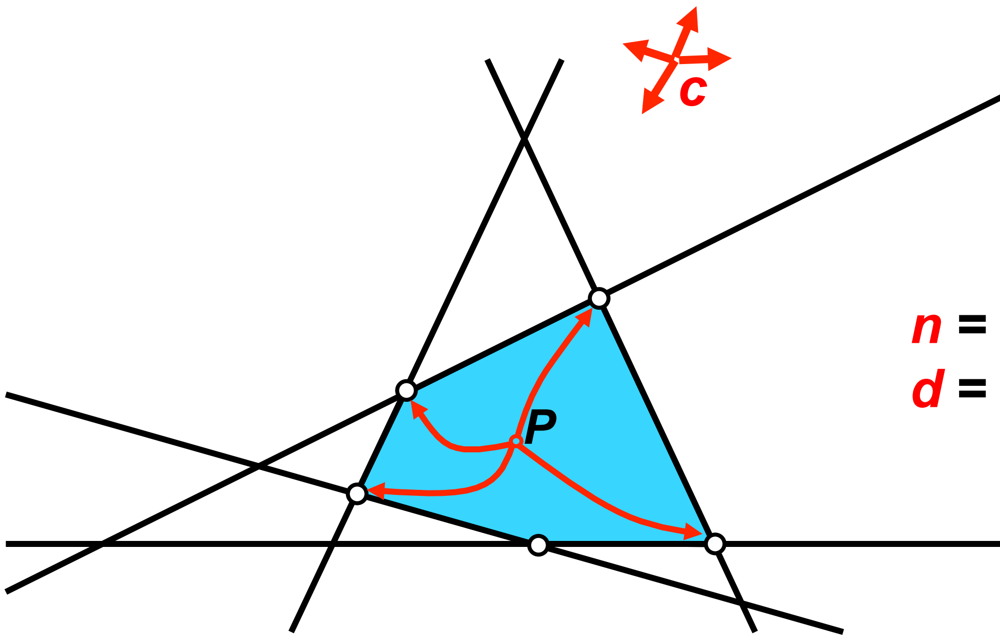
$n = 5$: *inequalities*

$d = 2$: *dimension*

$\lambda^{c}(P)$: total curvature of the primal central path of $\{ \max c^{T}x : x \in P \}$

$\lambda(P)$: *largest* total **curvature** $\lambda^{c}(P)$ *over of all possible* c

linear optimization : diameter and curvature



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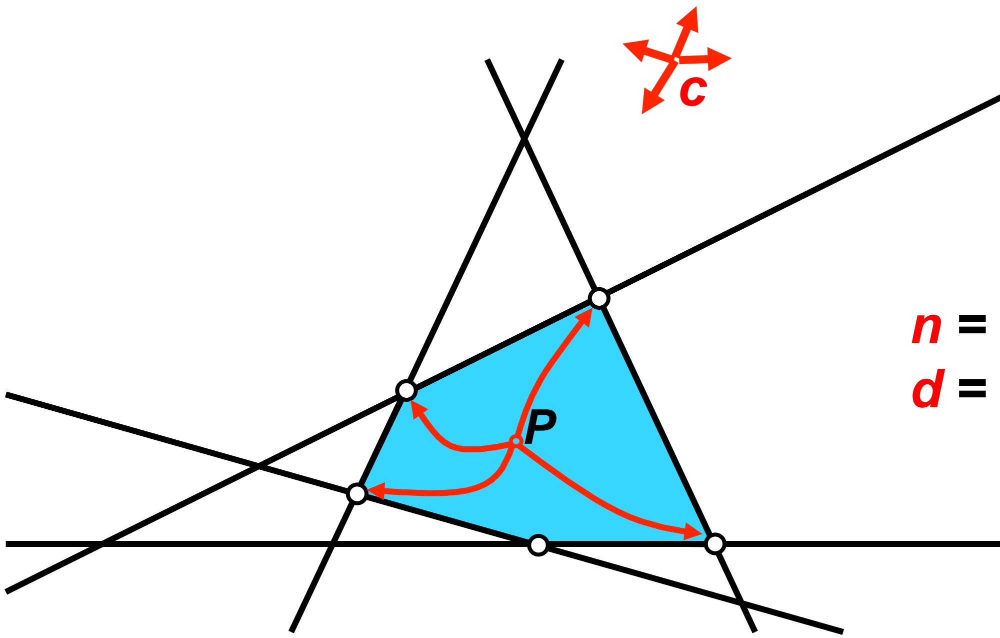
Continuous analogue of Hirsch Conjecture: $\lambda(P) = O(n)$
(Deza-Terlaky-Zinchenko 2008)

❖ Dedieu-Shub 2005 hypothesis : $\lambda(P) = O(d)$

linear optimization : diameter and curvature



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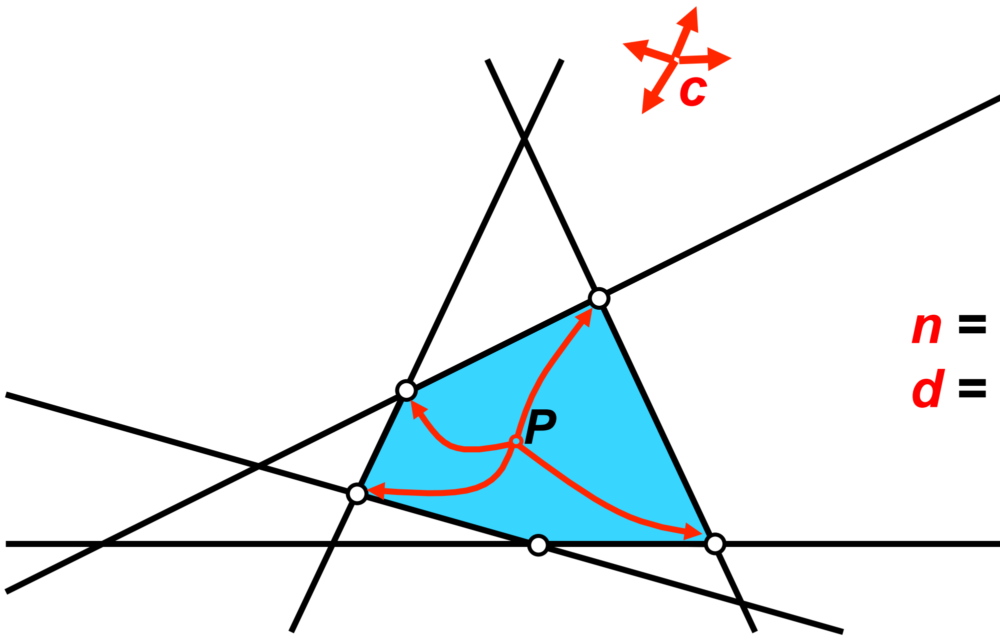
- Dedieu-Shub 2005 hypothesis : $\lambda(P) = O(d)$
- Deza-Terlaky-Zinchenko 2008 : polytope such that: $\lambda(P) = \Omega(2^{d})$

linear optimization : diameter and curvature



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Continuous analogue of Hirsch Conjecture: $\lambda(P) = O(n)$
(Deza-Terlaky-Zinchenko 2008)

- *disproved* by Allamigeon-Benchimol-Gaubert-Joswig 2014

linear optimization : diameter and curvature

Dedieu-Shub 2005 hypothesised $\lambda(\mathbf{P}) = O(d)$

Dedieu-Malajovich-Shub 2005 proved it is true *on average*

(de Loera-Sturmfels-Vinzant 2012)

Deza-Terlaky-Zinchenko 2008: ***P*** with exponential $\lambda(\boldsymbol{P})$ and $n = \Omega(2^d)$

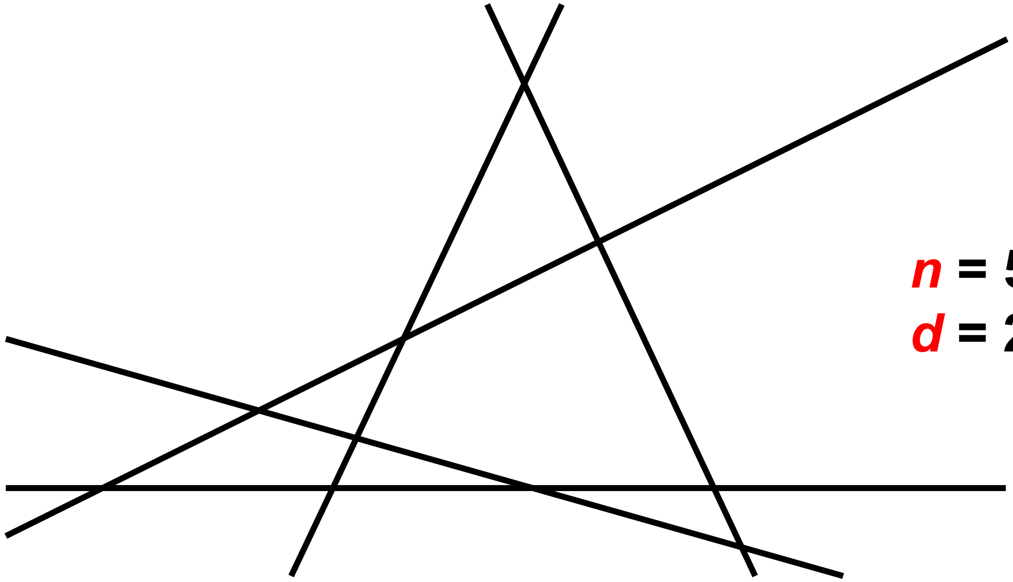
Continuous analogue of Hirsch Conjecture: $\lambda(P) = O(\mathrm{poly}(n,d))$

Allamigeon-Benchimol-Gaubert-Joswig 2014 : linear optimization instance ($2n \approx 3d$) for which central-path following methods require $\Omega(2^{d/2})$ iterations

⇒ ***path-following interior-point methods are not strongly polynomial***

Result obtained using ***tropical geometry***, which reduces the complexity analysis to a ***combinatorial*** problem

linear optimization : diameter and curvature

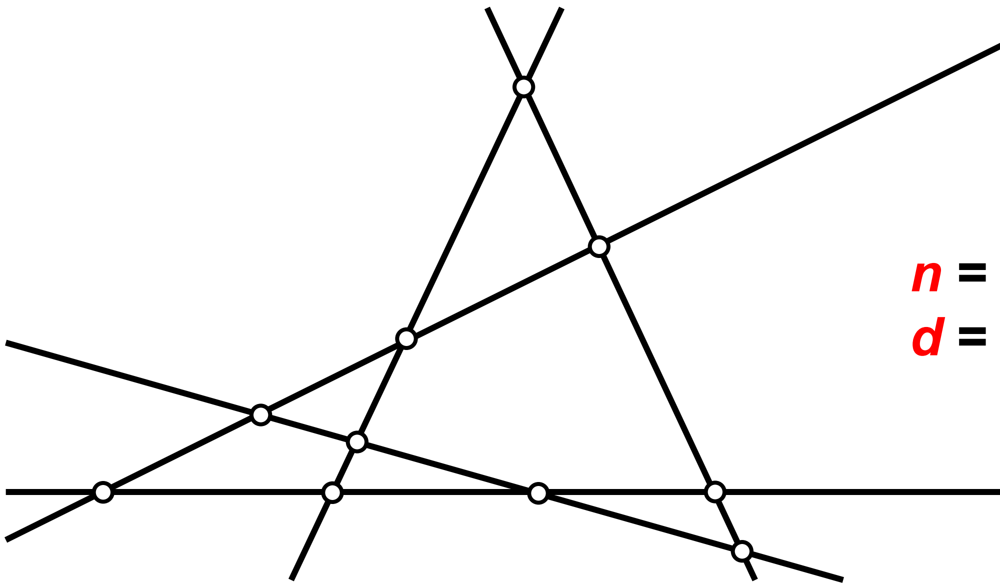


n **= 5** : *hyperplanes*

d **= 2** : *dimension*

Arrangement ***A*** defined by n **hyperplanes** in **dimension** d

linear optimization : diameter and curvature



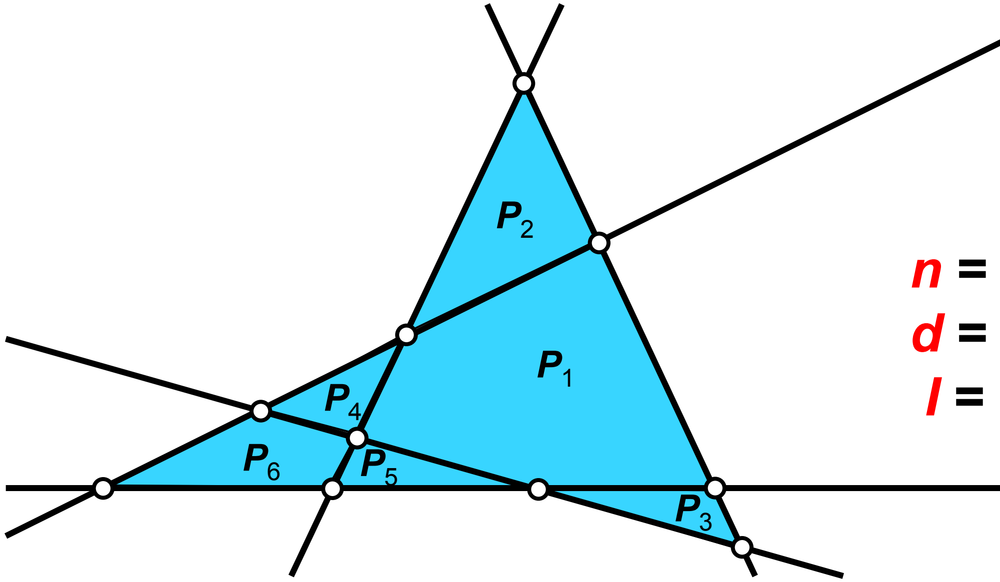
n **= 5** : *hyperplanes*

d **= 2** : *dimension*

Simple arrangement:

$n > d$ and any d hyperplanes **intersect** at a **unique distinct point**

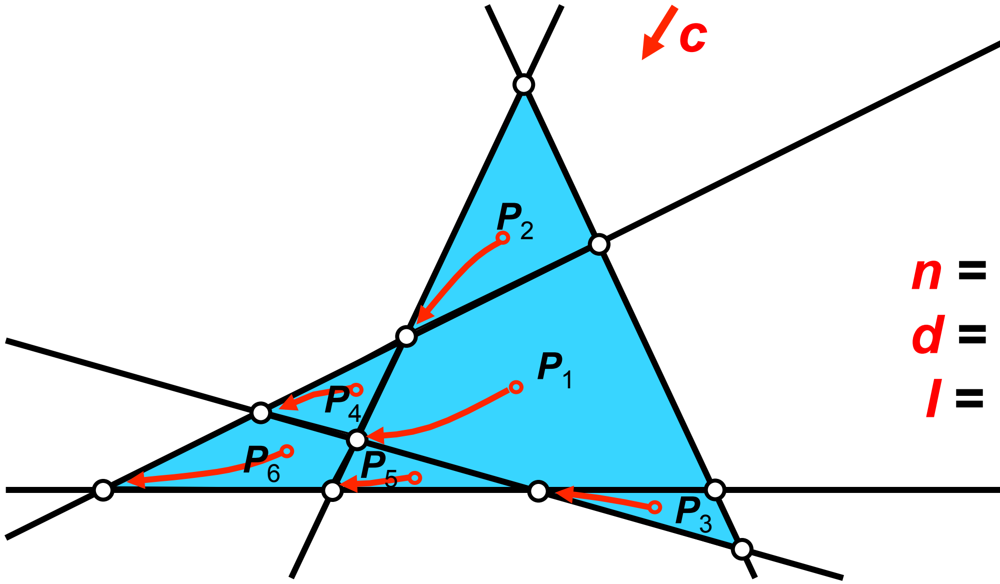
linear optimization : diameter and curvature



***n* = 5** : *hyperplanes*
***d* = 2** : *dimension*
***l* = 6** : *bounded cells*

For a simple arrangement, the number of **bounded cells *l* =** $\binom{n-1}{d}$

linear optimization : diameter and curvature



n **= 5** : *hyperplanes*

d **= 2** : *dimension*

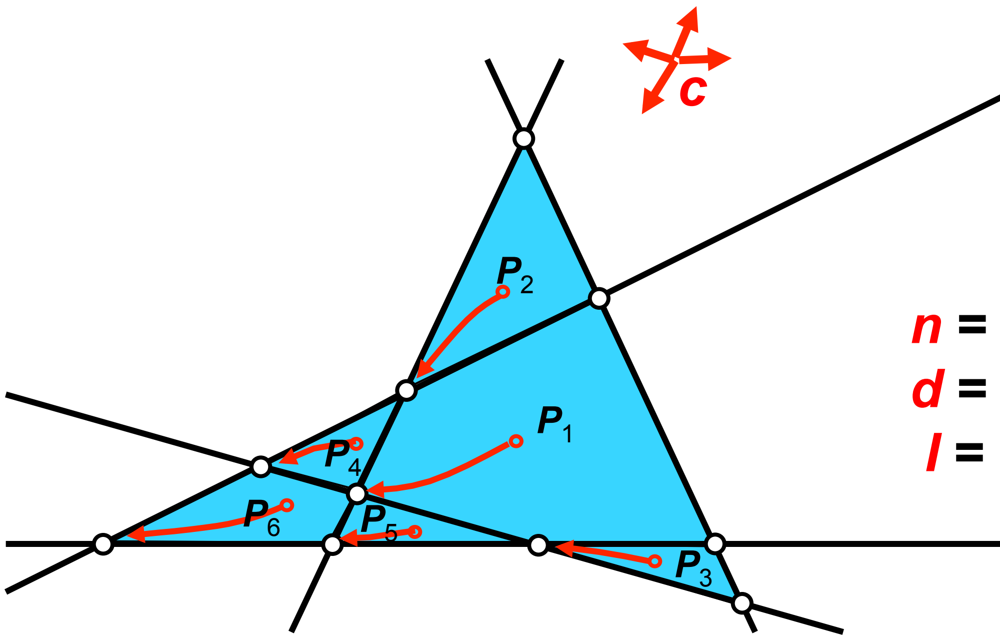
I **= 6** : *bounded cells*

$\lambda^c(A)$: average value of $\lambda^c(P_i)$ over the bounded cells P_i of A:

$$\lambda^c(A) = \frac{\sum_{i=1}^{i=I} \lambda^c(P_i)}{I} \qquad \text{with } I = \binom{n-1}{d}$$

❖ $\lambda^c(P_i)$: *redundant* inequalities count

linear optimization : diameter and curvature



$\boldsymbol{n}$ **= 5** : *hyperplanes*
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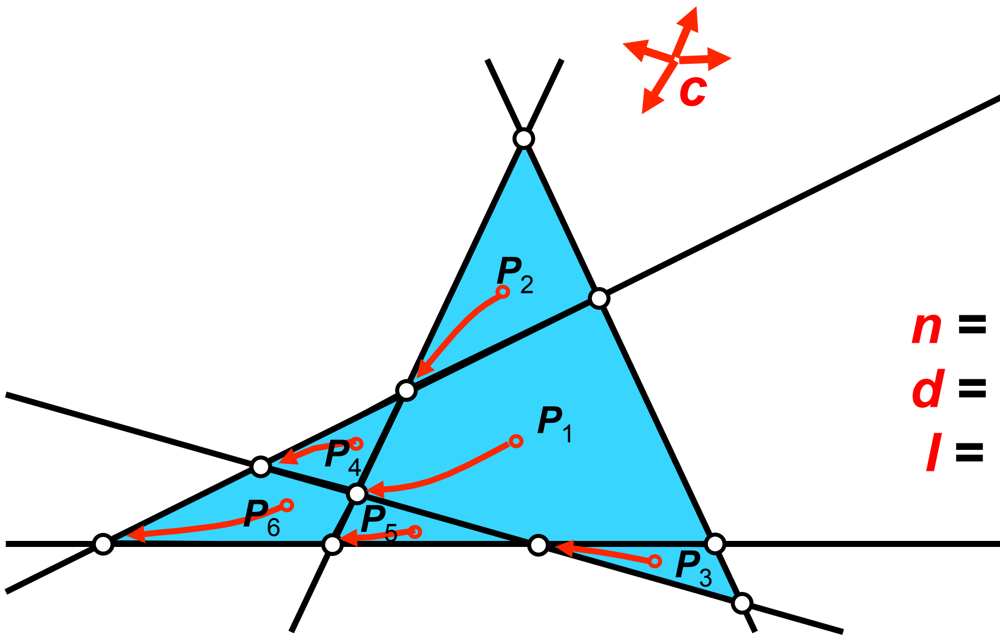
$\lambda^{c}(\boldsymbol{A})$: average value of $\lambda^{c}(\boldsymbol{P_i})$ over the bounded cells $\boldsymbol{P_i}$ of $\boldsymbol{A}$:

$\lambda(\boldsymbol{A})$: *largest* value of $\lambda^{c}(\boldsymbol{A})$ *over all possible* $\boldsymbol{c}$

linear optimization : diameter and curvature



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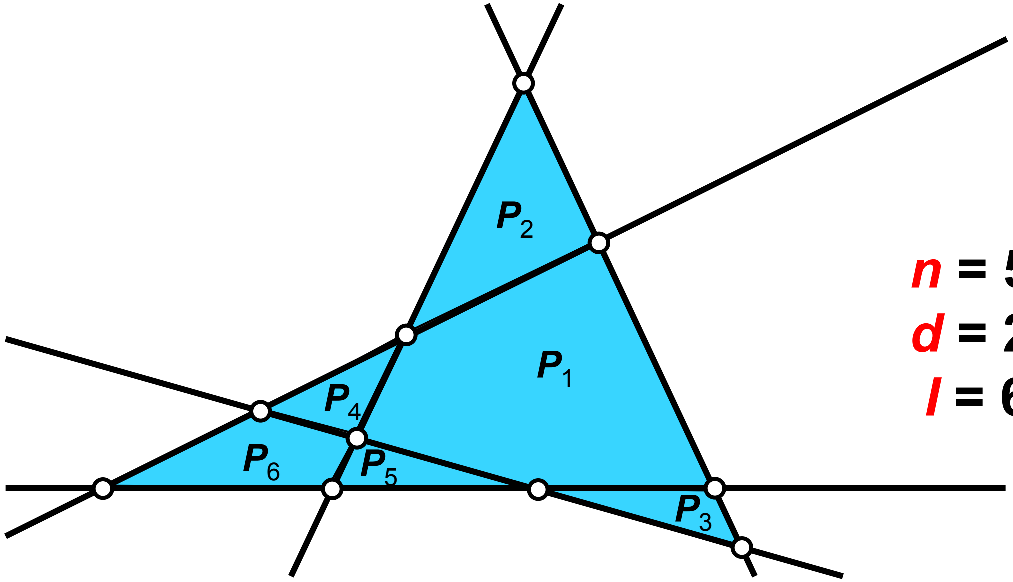
$\lambda(\boldsymbol{A})$: largest value of $\lambda^{c}(\boldsymbol{A})$ *over all possible* $\boldsymbol{c}$

Dedieu-Malajovich-Shub 2005: $\lambda(\boldsymbol{A}) \leq \boldsymbol{2\pi d}$

(de Loera-Sturmfels-Vinzant 2012)

❖ $\boldsymbol{A}$: *simple* arrangement

linear optimization : diameter and curvature



$\boldsymbol{n}$ **= 5** : *hyperplanes*
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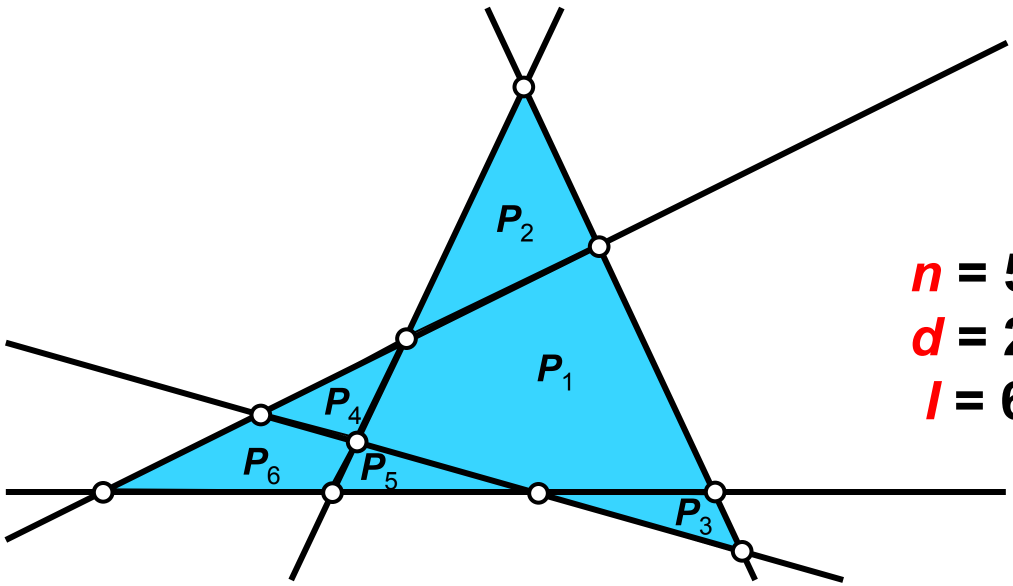
$\delta(\boldsymbol{A})$: average diameter of a bounded cell of $\boldsymbol{A}$:

❖ ***A*** : *simple* arrangement

linear optimization : diameter and curvature



***n* = 5** : *hyperplanes*
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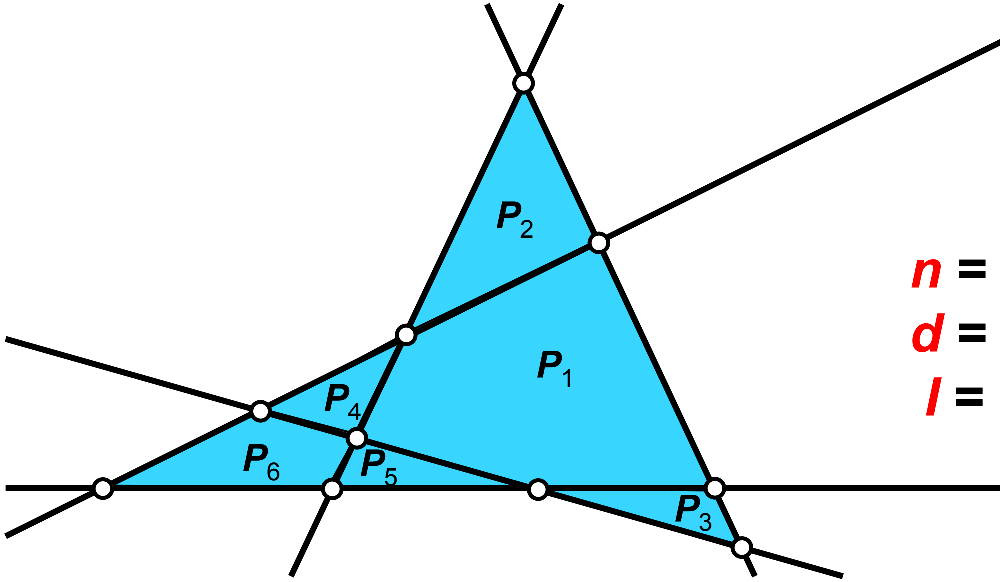
δ(***A***) : average diameter of a bounded cell of ***A***:

$$\delta(\boldsymbol{A}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I} \qquad \text{with } I = \binom{n-1}{d}$$

❖ δ(***A***): average diameter ≠ diameter of ***A***
ex: δ(***A***)= 1.333…

linear optimization : diameter and curvature



$n = 5$: *hyperplanes*
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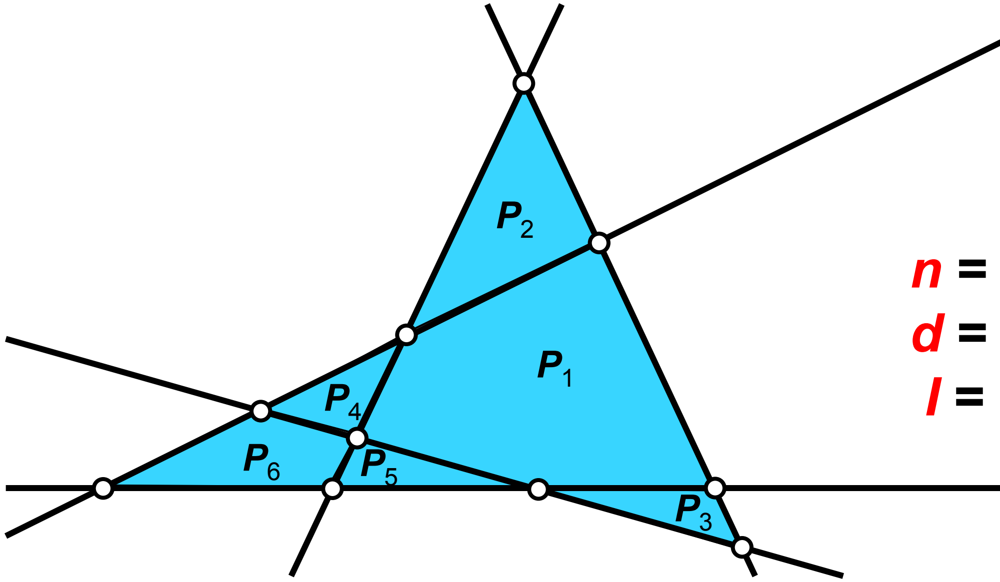
$\delta(A)$: average diameter of a bounded cell of A:

$$\delta(A) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I} \qquad \text{with } I = \binom{n-1}{d}$$

❖ $\delta(P_i)$: only *active* inequalities count

linear optimization : diameter and curvature



n **= 5** : *hyperplanes*
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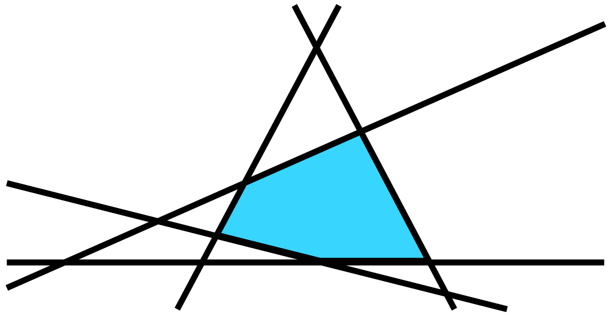
$\delta(\boldsymbol{A})$: average diameter of a bounded cell of ***A***:

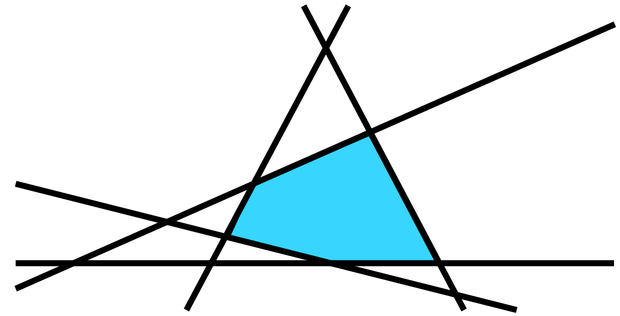
Conjecture : $\delta(\boldsymbol{A}) \leq d$
(Deza-Terlaky-Zinchenko 2008)

(*discrete analogue of Dedieu-Malajovich-Shub result*)

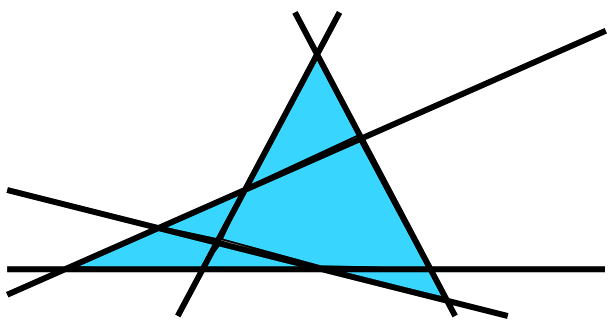
linear optimization : diameter and curvature



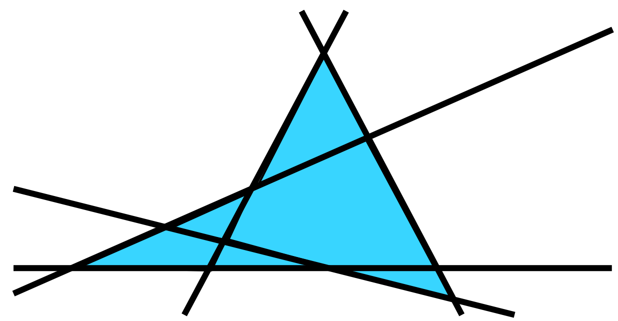
$\delta(P) \leq n - d$? Hirsch conjecture (1957)
Santos 2012



$\lambda(P) \leq 2\pi n$ Poly(n,d)? Deza-Terlaky-Zinchenko 2008
Allamigeon-Benchimol-Gaubert-Joswig 2014



$\delta(A) \leq d$? Deza-Terlaky-Zinchenko 2008



$\lambda(A) \leq 2\pi d$ Dedieu-Malajovich-Shub 2005

Terlaky-Mut 2014 : Sonnevend curvature

linear optimization : diameter and curvature

Hirsch bound $\delta(P) \le n - d$ **implies** $\delta(A) \le d \, \frac{n+1}{n-1}$

Hirsch conjecture *holds* for $d = 2$: $\delta(A) \le 2 \, \frac{n+1}{n-1}$

Hirsch conjecture *holds* for $d = 3$: $\delta(A) \le 3 \, \frac{n+1}{n-1}$

Larman 1970, Barnette 1974 $\delta(P) \le n 2^d / 12$
(Labbé-Manneville-Santos 2015)

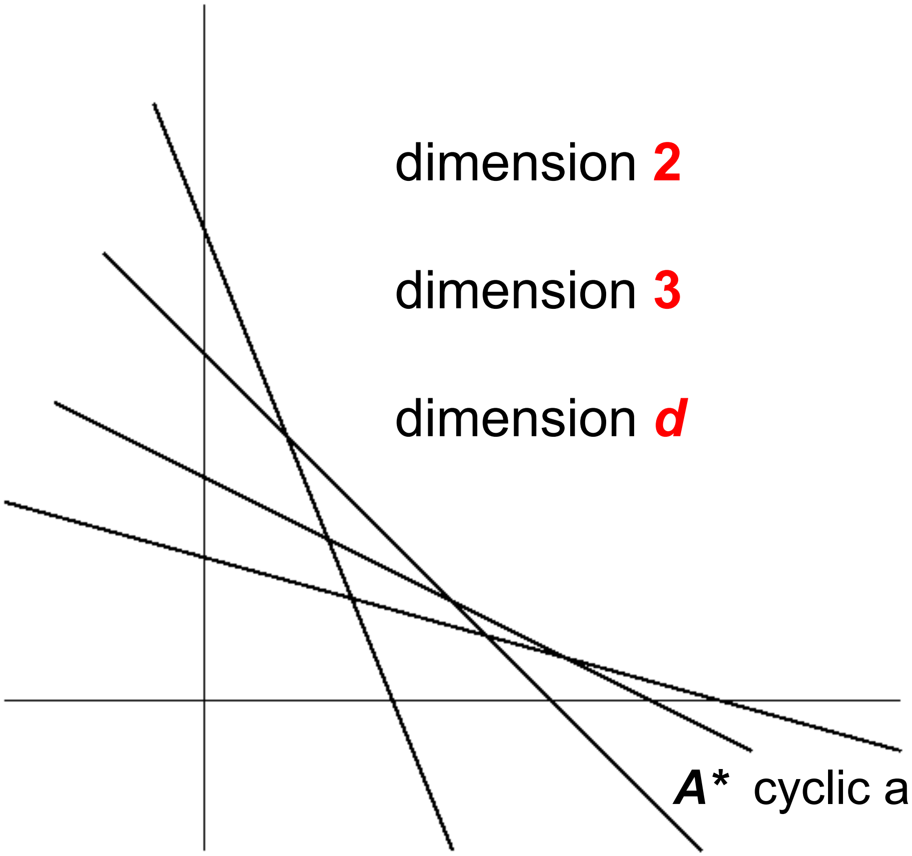
Kalai-Kleitman 1992 $\delta(P) \le n^{\log d + 2}$

Todd 2014 $\delta(P) \le (n - d)^{\log d}$

Sukegawa-Kitahara 2015 $\delta(P) \le (n - d)^{\log(d-1)}$

Sukegawa 2016, Mizuno-Sukegawa 2016
Borgwardt-de Loera-Finhold 2016 (Hirsch holds for transportation polytopes)
.....

linear optimization : diameter and curvature



dimension **2** $\delta(A) = \dfrac{2\lfloor n/2 \rfloor}{(n-1)(n-2)}$

dimension **3** $\delta(A)$ asympotically equal to **3**

dimension ***d*** $d \binom{n-d}{d} / \binom{n-1}{d} \leq \delta(A)$

Deza-Xie 2009

A* cyclic arrangement (mainly consists of cubical cells)

- Haimovich’s probabilistic analysis of shadow-vertex simplex method, Borgwardt 1987
- Forge-Ramírez Alfonsín 2001: counting *k*-face cells of ***A****

linear optimization : diameter and curvature

Diameter (*of a polytope*) :

lower bound for the number of iterations
for the **simplex method** (*pivoting methods*)

lower bound : $(1+\varepsilon)\,(n-d)$ **upper bound**: $(n-d)^{\log d}$

Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iteration
for *central path following* **interior point methods**

lower bound : $\Omega(2^{d/2})$ with $2n \approx 3d$ **upper bound**: $2\pi d \binom{n-1}{d}$

Allamigeon-Benchimol-Gaubert-Joswig 2014 **exponential lower bound** for $\lambda(P)$ contrasts with the belief that a **polynomial upper bound** for $\delta(P)$ might exist, e.g. $\delta(P) \leq d\,(n-d)/2$

linear optimization : diameter and curvature

Δ(***d***,***n***) : largest diameter over all ***d***-dimensional polytopes with ***n*** facets

Δ(***d***,***n***)		***n – d***				
		4	5	6	7	8
d	4	4	5	5	[6,7]	7+
	5	4	5	6	[7,9]	7+
	6	4	5	[6,7]	[7,9]	8+
	7	4	5	[6,7]	[7,10]	8+

Δ(4,10) = **5**, Δ(5,11) = **6** Goodey 1972

linear optimization : diameter and curvature

$\Delta(d,n)$: largest diameter over all d-dimensional polytopes with n facets

$\Delta(d,n)$		$n-d$ = 4	5	6	7	8
d	4	4	5	5	6	7+
	5	4	5	6	[7,8]	7+
	6	4	5	6	[7,9]	8+
	7	4	5	6	[7,10]	8+

$\Delta(4,11) = \Delta\,(6,12) = 6$ Bremner-Schewe 2011

linear optimization : diameter and curvature

$\Delta(d,n)$: largest diameter over all d-dimensional polytopes with n facets

$\Delta(d,n)$		$n-d$				
		4	5	6	7	8
d	4	4	5	5	6	7
	5	4	5	6	7	[7,9]
	6	4	5	6	[7,8]	[8,11]
	7	4	5	6	[7,9]	[8,12]

$\Delta(4,12) = \Delta\,(5,12) = 7$ Bremner-Deza-Hua-Schewe 2013

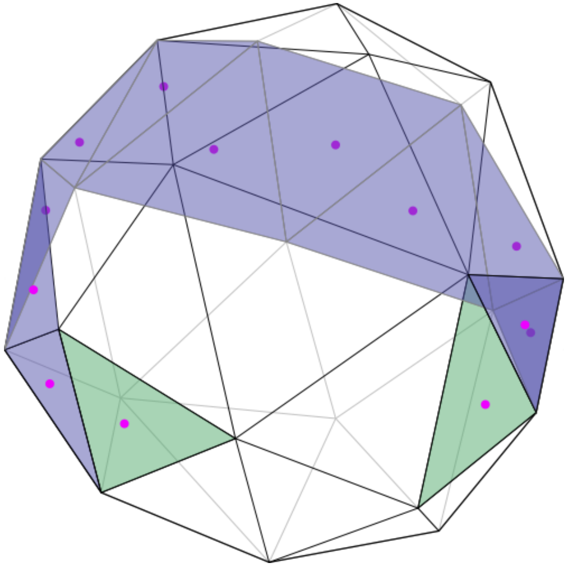
linear optimization : diameter and curvature

$\Delta(d,n)$: largest diameter over all d-dimensional polytopes with n facets

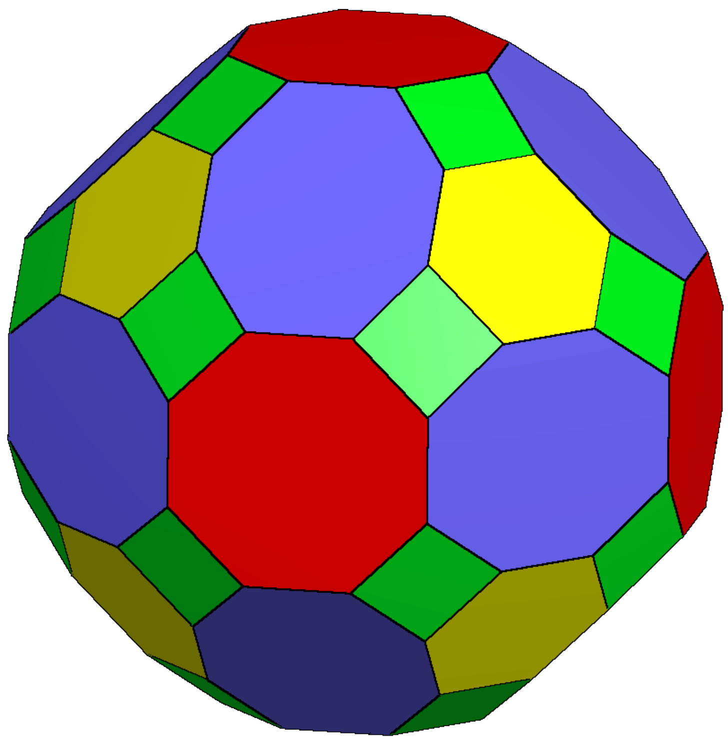
Characterize all combinatorial types of paths of length ℓ

Find necessary conditions for a (chirotope of a) polytope to admit an embedding of a ℓ-path on its boundary (without shortcuts)

If *no* such (chirotope of a) polytope exists: $\Delta(d, n) \neq \ell$



Algorithmic and geometric aspects of combinatorial and continuous optimization



Antoine Deza

McMaster University

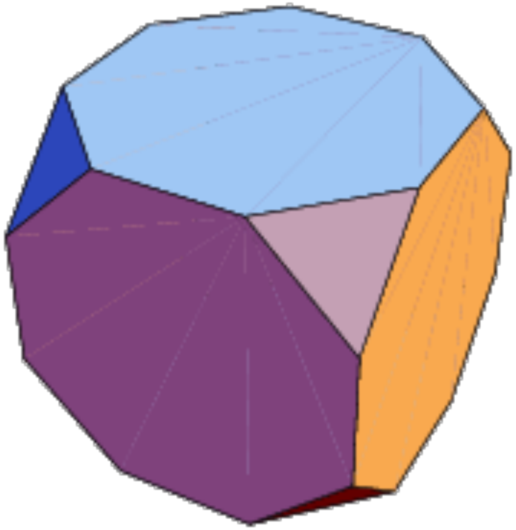
lattice polytopes with large diameter

lattice (d,k)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope P : smallest number such that **any two vertices** of P can be connected by a **path with at most $\delta(P)$ edges**

$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

ex. $\delta(3,3) = 6$ and is achieved
by the ***truncated cube***



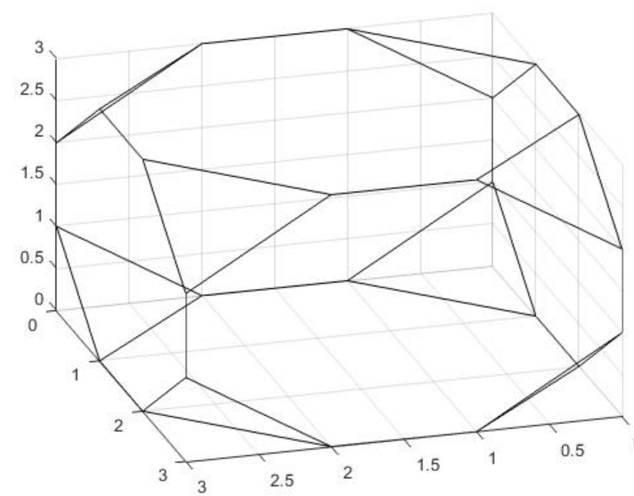
lattice polytopes with large diameter

lattice (***d***,***k***)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope ***P*** : smallest number such that **any two vertices** of ***P*** can be connected by a **path with at most** $\delta(P)$ **edges**

$\delta(d,k)$: largest diameter over all **lattice** (***d***,***k***)-polytopes

ex. $\delta(3,3) = 6$ and is achieved by the ***truncated cube***



lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

$\delta(d,1) = d$	[Naddef 1989]
$\delta(2,k) = O(k^{2/3})$	[Balog-Bárány 1991]
$\Rightarrow \delta(d,k) = \Omega(k^{2/3}\, d)$	[Del Pia-Michini 2016]
$\delta(d,k) \leq kd$	[Kleinschmid-Onn 1992]
$\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k)$	[Thiele 1991] [Acketa-Žunić 1995]
$\delta(d,2) = \lfloor 3d/2 \rfloor$	[Del Pia-Michini 2016]
$\delta(d,k) \leq kd - \lceil d/2 \rceil$ for $k \geq 2$	[Del Pia-Michini 2016]

lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	?	?	?	?	?	?
	4	4	6	?	?	?	?	?	?	?
	5	5	7	?	?	?	?	?	?	?

$\delta(d,k) = \Omega(k^{2/3}\, d)$

[Del Pia-Michini 2016]

lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	**4**	4	5	6	6	7	8
	3	3	4	6	**7+**	**9+**	?	?	?	?
	4	4	6	**8+**	**10+**	?	?	**16+**	?	?
	5	5	7	**10+**	?	**15+**	?	?	?	**25+**

$\delta(d,k) = \Omega(k^{2/3}\, d)$ [Del Pia-Michini 2016]

$\delta(d,k) \geq (k+1)d/2$ for many d and k [Deza-Manoussakis-Onn 2016]

convex matroid optimization

Motivation : convex matroid optimization [Melamed-Onn 2014]

The optimal solution of max { $\mathbf{f}(\mathbf{W}x)$: $x \in \mathbf{S}$} is attained at a vertex of the projection integer polytope in $\mathbb{R}^d$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^n$ (in the talk $\mathbf{S} \in \{0,1\}^n$)
W : integer d x n matrix (in the talk **W** is mostly {0,1}-valued)
f : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Q. What is the maximum number $\mathbf{v}(d,n)$ of vertices of conv(**WS**) when $\mathbf{S} \in \{0,1\}^n$ and **W** is a {0,1}-valued d x n matrix ?

Obviously $v(d,n) \leq |\mathbf{WS}| = O(n^d)$
In particular $v(2,n) = O(n^2)$, and $v(2,n) = \Omega(n^{0.5})$

convex matroid optimization

Motivation : convex matroid optimization [Melamed-Onn 2014]

S : set of feasible point $\in \mathbb{Z}^n$ (in the talk **S** $\in \{0,1\}^n$)
W : integer ***d*** x ***n*** matrix (in the talk **W** is mostly {0,1}-valued)
f : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Assume **S** in $\{0,1\}^n$ is a ***matroid*** of order ***n***; that is, the set of indicating vectors of bases of a matroid with ground set {1,…,***n***}

Given a matroid **S** of order ***n***, {0,1}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,1) of vertices of conv(**WS**) is independent of ***n*** and **S**

Given a matroid **S** of order ***n***, {0, ±1,…,±***p***}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

convex matroid optimization

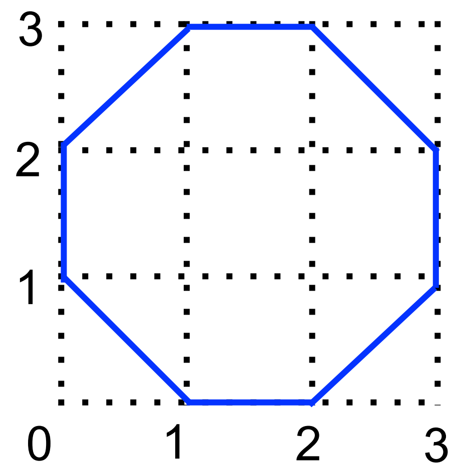
Motivation : convex matroid optimization [Melamed-Onn 2014]

Given a matroid **S** of order ***n***, {0,1}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,1) of vertices of conv(**WS**) is independent of ***n*** and **S**

Example : the maximum number **m**(2,1) of vertices of a planar projection conv(**WS**) of matroid **S** by a binary matrix **W** is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{S} = U(3,8) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



conv(**WS**)

primitive lattice polytopes

given a set *G* of ***m*** vectors (generators)

Zonotope *Z*(G) generated by *G* : convex hull of the 2^m ***signed*** sums of the ***m*** vectors in *G*

Minkowski sum *H*(*G*) : convex hull of the 2^m sums of the ***m*** vectors in *G*

Primitive lattice polytopes: Minkowski sum generated by ***short integer*** vectors which are ***pairwise linearly independent***

$$H_q(d,p) : H(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$$

$$Z_q(d,p) : Z(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$$

$x \succeq 0$: first nonzero coordinate of *x* is nonnegative

up to translation *Z*(*G*) is the image of *H*(*G*) by an homothety of factor 2

primitive lattice polytopes

(generalization of the permutahedron of type B_d)

$$H_q(d,p) : H(x \in \mathbf{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$$

$$H_q(d,p)^+ : H(x \in \mathbf{Z}^d_+ : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$$

(H^+ : ***positive*** primitive lattice polytope)

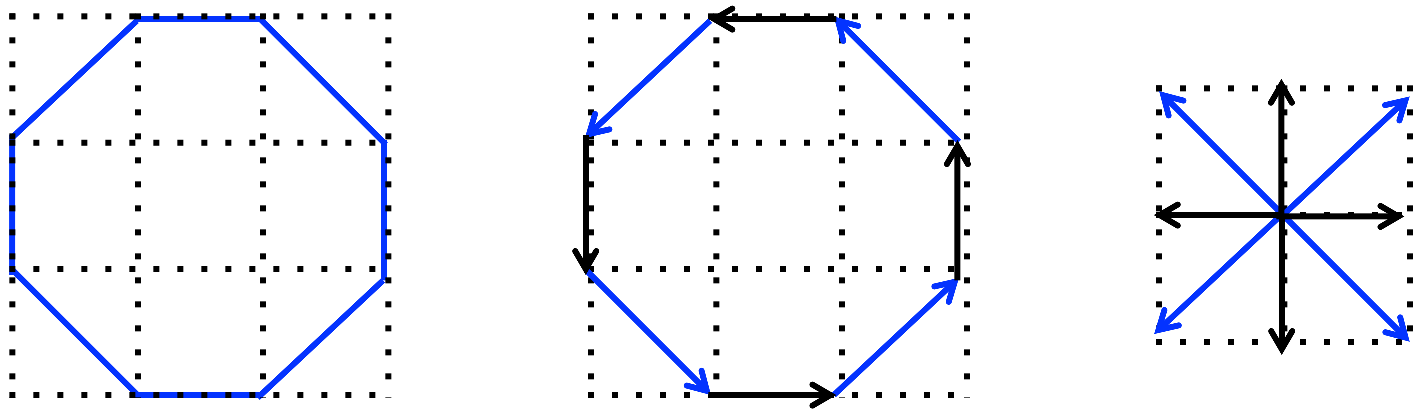
- $H_q(d,1)$: $[0,1]^d$ cube for finite q
- $H_1(3,2)$: truncated cuboctahedron (great rhombicuboctahedron)
- $H_\infty(3,1)$: truncated small rhombicuboctahedron
- $Z_1(d,2)$: permutahedron of type B_d
- $Z_1(d,2)^+$: Minkowski sum of the permutahedron and $[0,1]^d$

$\Rightarrow$ $Z_q(d,p)$ is invariant under permutations and sign flips

lattice polygons with large diameter

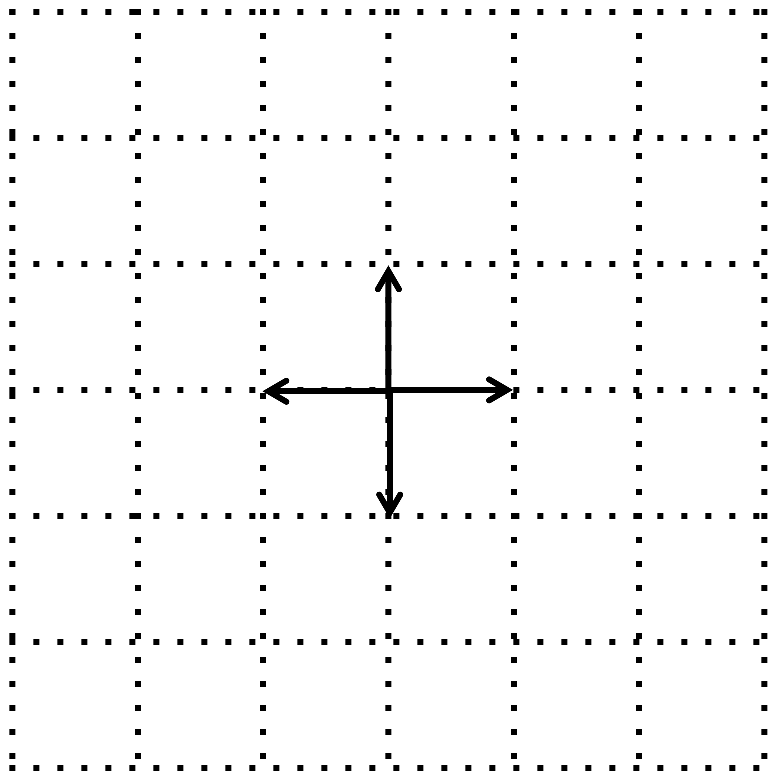
Q. What is $\delta(2,\boldsymbol{k})$: largest diameter of a polygon which vertices are drawn form the $\boldsymbol{k}$ x $\boldsymbol{k}$ grid?

A polygon can be associated to a set of vectors (*edges*) summing up to zero, and without a pair of positively multiple vectors



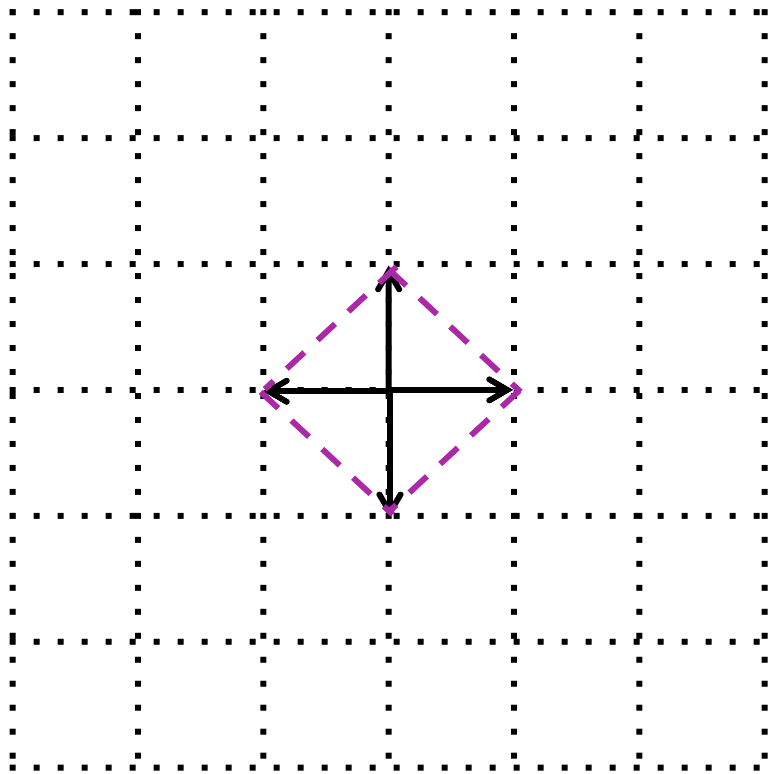
$\delta(2,3) = 4$ is achieved by 8 vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

lattice polygons with large diameter



$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

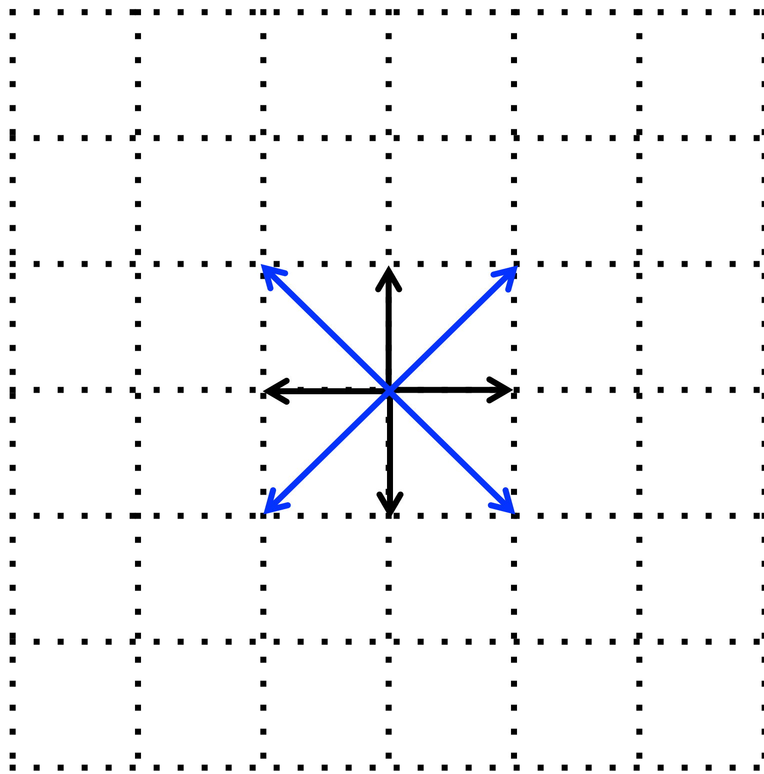
lattice polygons with large diameter



$||x||_1 \leq \mathbf{1}$

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

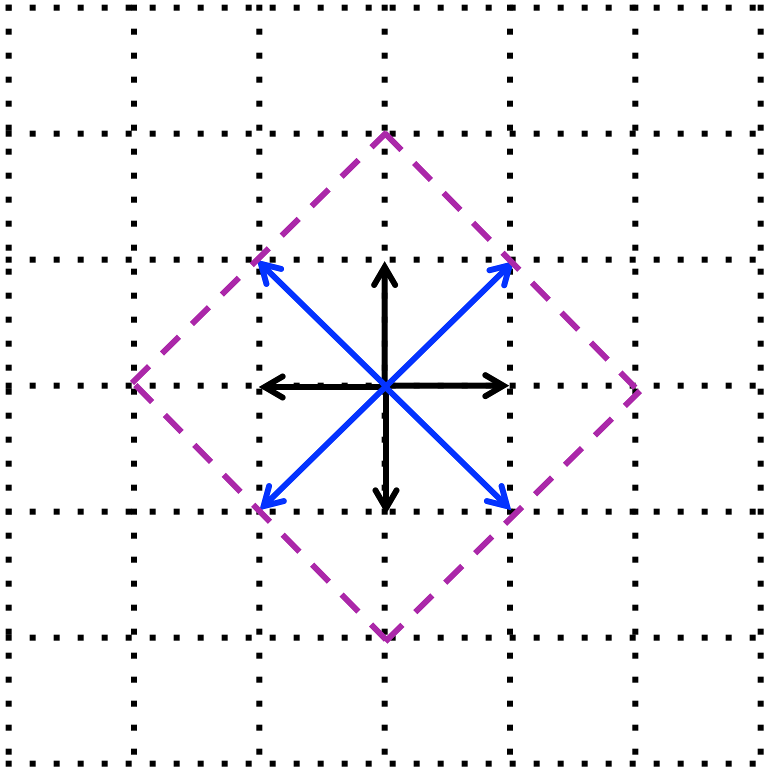
lattice polygons with large diameter



$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

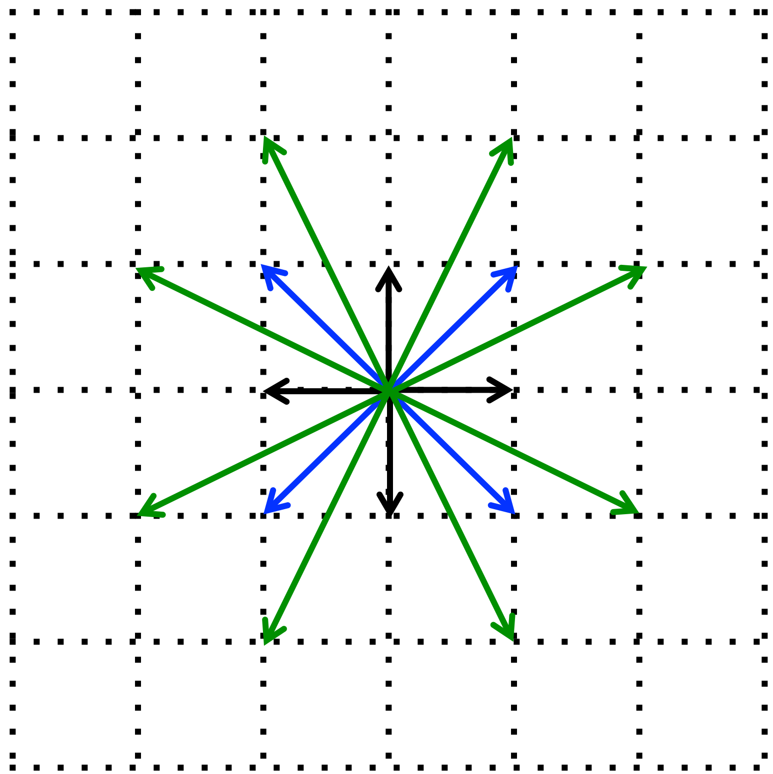
lattice polygons with large diameter



$||x||_1 \leq 2$

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$
$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

lattice polygons with large diameter

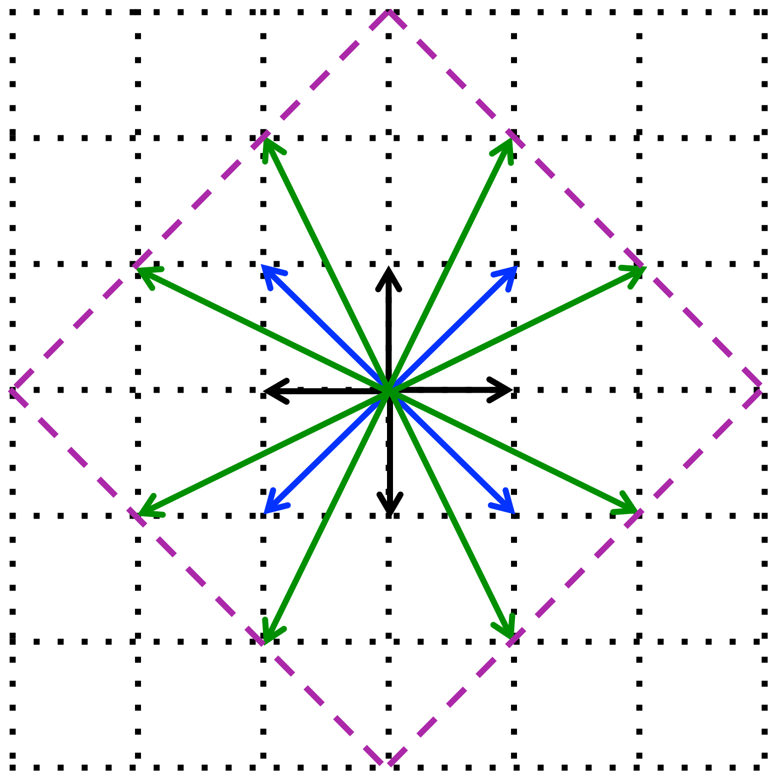


$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

$\delta(2,9) = 8$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$, $(\pm 1,\pm 2)$, $(\pm 2,\pm 1)$

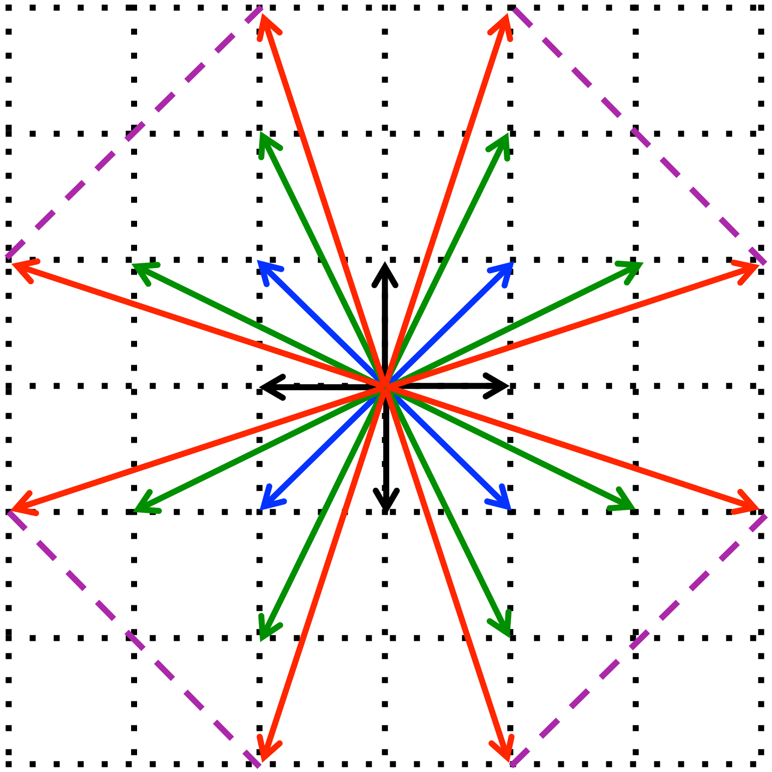
lattice polygons with large diameter



$||x||_1 \leq 3$

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$
$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$
$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

lattice polygons with large diameter



$||x||_1 \leq 4$

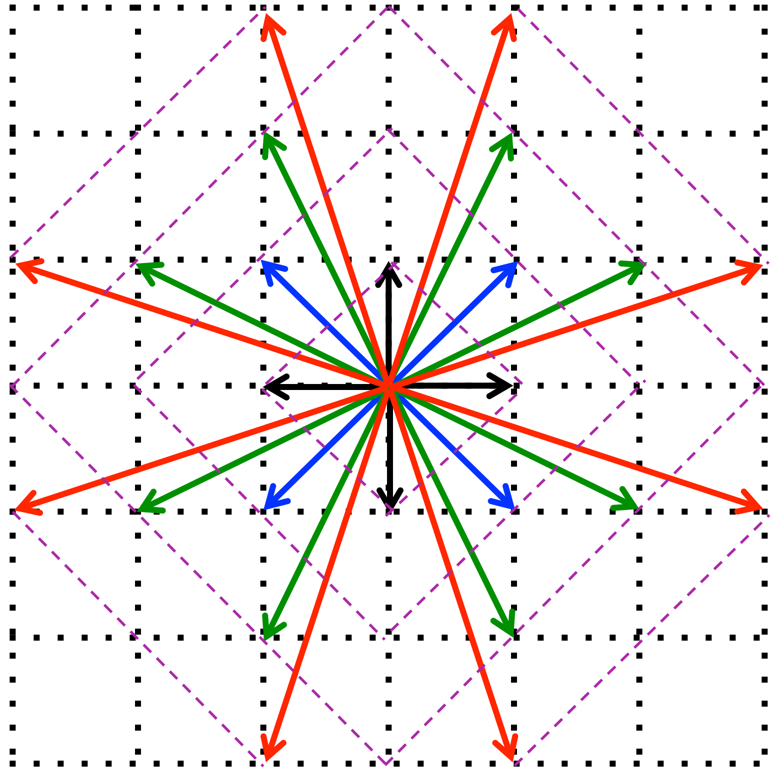
$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

$\delta(2,17) = 12$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$, $(\pm1,\pm3)$, $(\pm3,\pm1)$

lattice polygons with large diameter



$||x||_1 \leq p$

$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

$\varphi(p)$: ***Euler totient function*** counting positive integers less or equal to p relatively prime with p

$\varphi(1) = \varphi(2) = 1, \varphi(3) = \varphi(4) = 2, \ldots$

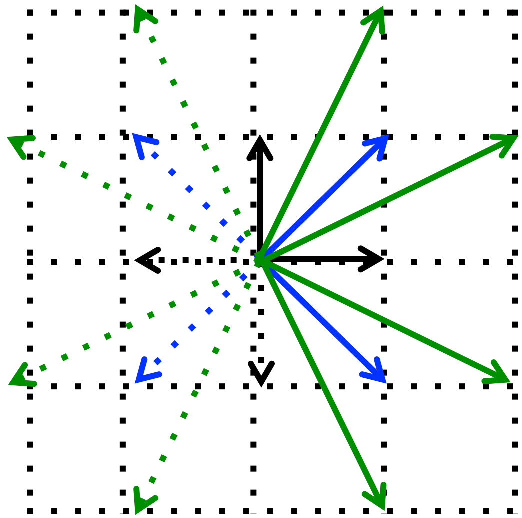
lattice polygons with large diameter

$\delta(2,k)$		*k*								
		1	2	3	4	5	6	7	8	9
	p	1		2						3
	v	4	6	8	8	10	12	12	14	16
	δ	2	3	4	4	5	6	6	7	8

$$\delta(2,\boldsymbol{k}) = 2\sum_{i=1}^{p}\varphi(i) \text{ for } \boldsymbol{k} = \sum_{i=1}^{p} i\varphi(i)$$

φ(***p***) : ***Euler totient function*** counting positive integers less or equal to ***p*** relatively prime with ***p***
φ(1) = φ(2) = 1, φ(3) = φ(4) = 2,…

lattice polygons with large diameter



$||x||_1 \leq p$

$H_q(d,p)$: Minkowski sum generated by $\{x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0\}$

$H_1(2,p)$ has diameter $\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by (1,0), (0,1), (1,1), (1,-1) (fits, *up to translation*, in 3x3 grid)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

graphical zonotopes

Graphical zonotope : Minkowski sum of segments $[e_i, e_j]$ for all edges {i,j} of a given graph. Ex. permutahedron : graphical zonotope of the complete graph on ***d*** nodes

For ***k*** ≤ ***d***, graphical zonotope $Z^*(d,k)$ of ***circulant graph*** of degree ***k***-1 on ***d*** nodes, with a loop at each node (Minkowski sum of a cube) has :

dimension ***d***
grid size embedding ***k***
diameter $(k+1)d/2$

Ex. $H_1(d,2)^+$: lattice (d,d)-polytope with diameter $d(d+1)/2$

lattice polytopes with large diameter

For $k \leq d$, graphical zonotope $Z^*(d,k)$ of the circulant graph of degree k-1 on d nodes, with a loop at each node

$\delta(Z^*(d,k)) = (k+1)d/2$ with $k \leq d$

$\delta(H_1(d,2))$ lattice $(d,2d-1)$-polytope with diameter d^2 with $k = 2d - 1$

$\delta(2,k) = \delta(H_1(2,p))$ for infinitely many k

$\delta(2,k) = \delta(H(G))$ for G subset of the generators of $H_1(2,p)$ for some p

$\delta(d,1) = H_1(d,1)$

$\delta(d,2) = \delta(H(G))$ for G subset of the generators of $H_1(d,2)$

$\delta(3,3) = \delta(H_1(3,2)^+)$

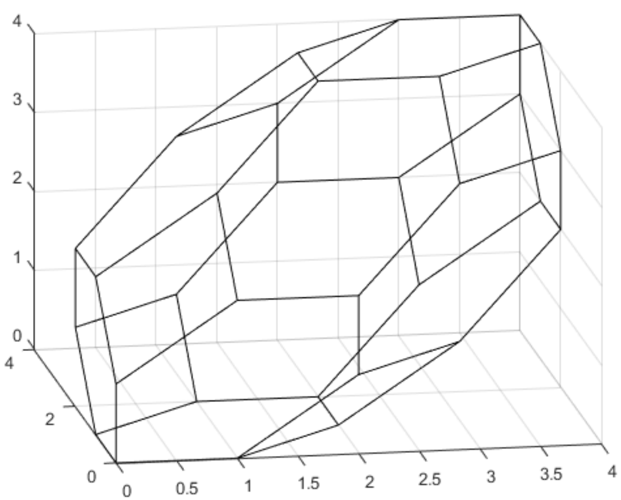
$H_q(d,p)$: Minkowski sum of $\{x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0\}$

convex matroid optimization

m(d,1): maximum number of vertices of conv(**WS**) over matroid **S** of order n, and {0,1}-valued d x n matrix **W**

Deza-Manoussakis-Onn 2016 : $|\, H_\infty(d,p)^+ \,| \le \mathbf{m}(d,p) \le |\, H_\infty(d,p) \,|$

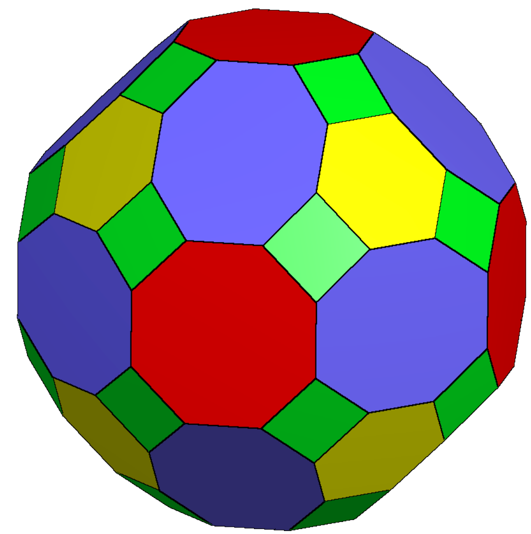
$$\Rightarrow \ |\, H_\infty(d,1)^+ \,| \le \mathbf{m}(d,1) \le |\, H_\infty(d,1) \,|$$



32 $\le$ **m**(3,1) $\le$ **96**

370 $\le$ **m**(4,1) $\le$ **5 376**

11 292 $\le$ **m**(5,1) $\le$ **1 981 440**



$H_\infty(3,1)^+$

$H_\infty(3,1)$: truncated small rhombicuboctahedron

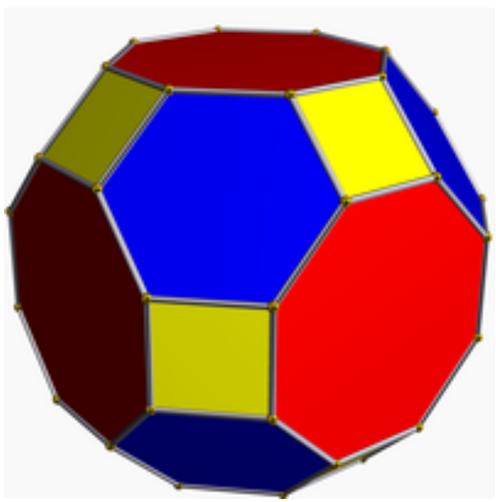
| ***P*** | *: number of vertices of* ***P***

convex matroid optimization

m(***d***,1): maximum number of vertices of conv(**WS**) over matroid **S** of order ***n***, and {0,1}-valued ***d*** x ***n*** matrix **W**

Deza-Manoussakis-Onn 2016 : $|\, H_\infty(d,p)^+ \,| \le \mathbf{m}(d,p) \le |\, H_\infty(d,p) \,|$

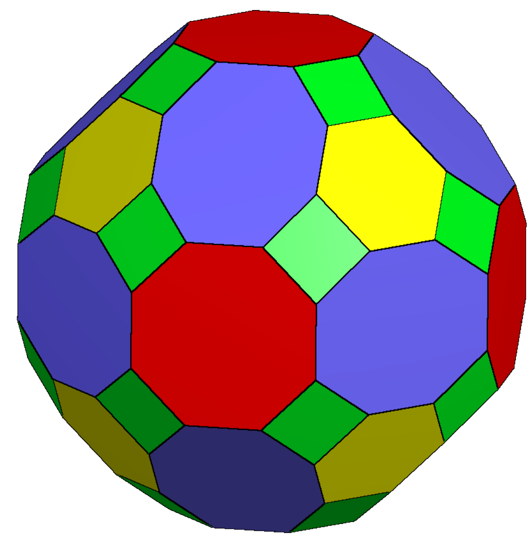
$$\Rightarrow \ |\, H_\infty(d,1)^+ \,| \le \mathbf{m}(d,1) \le |\, H_\infty(d,1) \,|$$



$32 \le \mathbf{m}(3,1) \le 96$

$370 \le \mathbf{m}(4,1) \le 5\,376$

$11\,292 \le \mathbf{m}(5,1) \le 1\,981\,440$



truncated cuboctahedron (great rhombicuboctahedron)

$H_\infty(3,1)$: truncated small rhombicuboctahedron

convex matroid optimization

m(d,1): maximum number of vertices of conv(**WS**) over matroid **S** of order n, and {0,1}-valued d x n matrix **W**

$$d2^{d} \leq \mathbf{m}(d,1) \leq 2\sum_{i=0}^{d-1}\binom{(3^{d}-3)/2}{i}$$

24 ≤ **m**(3) ≤ 158
64 ≤ **m**(4) ≤ 19 840

Melamed-Onn 2014

convex matroid optimization

m(*d*,1): maximum number of vertices of conv(**WS**) over matroid **S** of order *n*, and {0,1}-valued *d* x *n* matrix **W**

$d2^d \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$

$2d! \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i} - 2\binom{(3^{d-1}-3)/2}{d-1}$

24 ≤ **m**(3) ≤ 158
64 ≤ **m**(4) ≤ 19 840

Melamed-Onn 2014

48 ≤ **m**(3) ≤ 96
672 ≤ **m**(4) ≤ 5 376

Deza-Manoussakis-Onn 2016

lattice polytopes with large diameter

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

Conjecture (Deza-Manoussakis-Onn 2016)

- $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of lattice vector
- $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$

$\Rightarrow \delta(d,d) = d(d+1)/2$
$\Rightarrow \delta(d,2d-1) = d^2$

Conjecture holds for all (d,k) such that $\delta(d,k)$ is known

related questions

Soprunov-Soprunova 2016. ***Minkowski length*** $L(\boldsymbol{P})$ of a lattice polytope $\boldsymbol{P}$: largest number of laltice segments which Minkowski sum is contained in $\boldsymbol{P}$

denote $L(\{0,1,\ldots,\boldsymbol{k}\}^{\boldsymbol{d}})$ by $L(\boldsymbol{d},\boldsymbol{k})$

$L(\boldsymbol{d},1) = \delta(H_{\mathbf{1}}(\boldsymbol{d},1)) = \delta(\boldsymbol{d},1)$

$L(\boldsymbol{d},2) = \delta(H(G)) = \delta(\boldsymbol{d},2)$ G subset of generators of $H_{\mathbf{1}}(\boldsymbol{d},2)$

$L(2,\boldsymbol{k}) = \delta(H(G)) = \delta(2,\boldsymbol{k})$ G subset of generators of $H_{\mathbf{1}}(2,\boldsymbol{p})$ for some $\boldsymbol{p}$

$L(3,3) = \delta(H_{\mathbf{1}}(3,2)^{+}) = \delta(3,3)$

$L(\boldsymbol{d},\boldsymbol{d}) = \delta(H_{\mathbf{1}}(\boldsymbol{d},2)^{+})$

$L(\boldsymbol{d},2\boldsymbol{d}-1) = \delta(H_{\mathbf{1}}(\boldsymbol{d},2))$

Sloane OEI sequences

$H_{\infty}(\boldsymbol{d},1)^{+}$ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till $\boldsymbol{d}$ =8)

$H_{\infty}(\boldsymbol{d},1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension d (determined till $\boldsymbol{d}$ =7)

complexity questions

Deza-Manoussakis-Onn 2016: for ***fixed*** positive ***integers*** $\boldsymbol{p}$ and $\boldsymbol{q}$, linear optimization over $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$ is polynomial-time solvable, even in ***variable*** dimension $\boldsymbol{d}$.

⇒ for ***fixed*** positive ***integers*** $\boldsymbol{p}$ and $\boldsymbol{q}$, the following problems are polynomial time solvable.

- ***extremality***: given $x \in \mathbb{Z}^{\boldsymbol{d}}$, decide if x is a vertex of $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$
- ***adjacency***: given $x_1,x_2 \in \mathbb{Z}^{\boldsymbol{d}}$, decide if $[x_1,x_2]$ is an edge of $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$
- ***separation***: given rational $\mathrm{y} \in \mathbb{R}^{\boldsymbol{d}}$, either assert $\mathrm{y} \in Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$, or find $h \in \mathbb{Z}^{\boldsymbol{d}}$ separating y from $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$, that is, satisfying $h^{\mathrm{T}}\mathrm{y} > h^{\mathrm{T}}\mathrm{x}$ for all $x \in Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$